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JOURNAL OF PURE MATHEMATICS

Volume 23, 2006

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G-146093

127/8 CUK-H004058-8-G146093

Jour. Pure Math., Vol. 23, 2006, pp. 1-13

(A, φ) COMPACTNESS AND SOME RESULTS AS AN APPLICATION

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ABSTRACT : The aim of this paper is to continue the unification of some concepts in topological spaces similar to compactness. The relations between the ($\mathcal{A}$, φ)-compactness, filters and ($\mathcal{A}$, φ)-closure operators are investigated. Some equivalence forms of quasi H-closed space and nearly compact space are given as an application.

Key words and pharases : Compactness, filter, convergence, operation, unification.
2000 AMS Classification : 54*A*20, 54*D*30.

1. INTRODUCTION

Various unifications by using operations and some similar concepts have been studied in [1, 3, 5, 6, 7, 8] such as compactness, countable-compactness, filter convergence and continuity. In this paper, the relations between the ($\mathcal{A}$, φ)-compactness, filters and ($\mathcal{A}$, φ)-closure operators are investigated. Some equivalent forms of quasi H-closed space and nearly compact space are given as an application.

2. PRELIMINARIES

Throughout this paper, by (X, τ) we will denote a topological space with no seperation properties assumed on X. Let A be a subset of a topological space (X, τ). $X \setminus A$ will denote the complement of A in X. The closure of A and the interior of A with respect to τ are denoted by clA and intA, respectively. A is said to be semi-open (preopen, regular open, α-open) if $A \subset cl(intA)$ $(A \subset int(clA)$, $A = int(clA)$, $A \subset int(cl(intA)))$. The complement of a semi-open (preopen, regular open, α-open) is said to be semi-closed (preclosed, regular closed, α-closed). Semi-closure and pre-closure are defined in a manner analogous to the corresponding concept of closure; scl and pcl will denote the semi-closure and the pre-closure operations. The θ-closure, δ-closure, θ-interior and δ-interior are defined in the following way and denoted by θcl, δcl, θint and δint, respectively.

Let A be a subset of X.

θclA = $\{x \in X$: for every open set U containing x, $\overline{U} \cap A \neq \phi\}$,

$\delta \mathrm{clA} = \{x \in X$: for every regular open set U containing, x, $U \cap A \neq \phi\}$,

$\theta \mathrm{intA} = \{x \in X$: there exists an open set U containing x such that $x \in U \subset \overline{U} \subset A\}$,

$\delta \mathrm{intA} = \{x \in X$: there exists a regular open set U containing x such that $x \in U \subset A\}$.

A is said to be θ-open (δ-open, θ-closed, δ-closed) if A = θintA (A = δintA, A = θclA, A = δclA), respectively.

Now, we give the following basic definitions.

Definition 2.1. *Let (X, τ) be a topological space. A mapping $\varphi : P(X) \to P(X)$ is called an operation on (X, τ) if*

i) $A^\circ \subset \varphi(A)$ *for all* $A \in P(X)$

ii) $\varphi(0) = 0$.

The class of all operations on a topological space (X, τ) will be denoted by $O(X, \tau)$. If not otherwise stated, all operations will be defined on (X, τ).

Definition 2.2. *Let (X, τ) be a topological space.*

i) *A partial órder "≤" on $O(X, \tau)$ is defined as $\varphi_1 \leq \varphi_2 \Leftrightarrow \varphi_1 (A)\ \varphi_2 (A)$ for each $A \in P(X)$.*

ii) *An operation $\varphi \in O(X, \tau)$ is called monotonous if $\varphi(A) \subset \varphi(B)$ whenever $A \subset B$ $(A, B \in P(X))$.*

Definition 2.3. *Let $\varphi \in O(X, \tau)$ and, $A, B \subset X$. A is called φ-open if $A \subset \varphi(A)$. Likewise B is called φ-closed if $X \setminus B$ is φ-open.*

Operation $\varphi^\sim \in O(X, \tau)$ is called the dual operation of φ if $\varphi^\sim(A) = X \setminus \varphi(X \setminus A)$ *for each* $A \in P(X)$.

Let $\varphi \in O(X, \tau)$, $\mathcal{U} \subset P(X)$, $x \in X$. We use the following notations.

$\varphi O(X) = \{U : U \subset X, U$ is φ-open$\}$,

$\varphi O(X, x) = \{U : U \in \varphi O(X), x \in U\}$,

$\mathcal{U}(x) = \{U : x \in U \in \mathcal{U}\}$,

$\mathcal{U}^c = \{U : X \setminus U \in \mathcal{U}\}$.

Note that for any operation $\varphi \in O(X, \tau)$, $\tau \subset \varphi O(X)$ and X and are both φ-open and φ-closed. If φ is monotonous then $\varphi O(X)$ is closed under arbitrary unions.

For the benefit of the reader we will recall necessary definitions and results in [6, 8].

Definition 2.4. [5] *Let* $X \in \mathcal{A} \subset P(X)$, $A \subset X$ *and* $\varphi \in O(X, \tau)$.

a) $x \in (\mathcal{A}, \varphi)\text{-}clA \Leftrightarrow$ *for each* $U \in \mathcal{A}(x)$, $\varphi(U) \cap A \neq \emptyset$.

b) $x \in (\mathcal{A}, \varphi)\text{-}intA \Leftrightarrow$ *there exists an* $U \in \mathcal{A}(x)$ *such that* $\varphi(U) \subset A$.

Definition 2.5. [5] *Let* $\varphi_1, \varphi_2 \in O(X, \tau)$, $A \subset X$.

a) $x \in \varphi_{1,2}intA \Leftrightarrow x \in (\varphi_1 O(X), \varphi_2)\text{-}intA$.

b) $x \in \varphi_{1,2}clA \Leftrightarrow x \in (\varphi_1 O(X), \varphi_2)\text{-}clA$.

c) A *is* $\varphi_{1,2}$*-open* $\Leftrightarrow A \subset \varphi_{1,2}intA$.

d) A *is* $\varphi_{1,2}$*-closed* $\Leftrightarrow \varphi_{1,2}clA \subset A$.

For $\varphi_1, \varphi_2 \in O(X, \tau)$, $\varphi_{1,2}O(X)$, $\varphi_{1,2}C(X)$ will stand for the family of $\varphi_{1,2}$-open sets and $\varphi_{1,2}$-closed sets, respectively.

Example 2.6. *Let* $\varphi_1 = int$, $\varphi_2 = intocl$, $\varphi_3 = int$ *and* $\varphi_4 = cl$. *Then,*

a) $x \in \varphi_{1,2}intA \Leftrightarrow x \in \delta\text{-}intA$,

b) $x \in \varphi_{3,4}intA \Leftrightarrow x \in \theta\text{-}intA$,

c) $x \in \varphi_{1,2}clA \Leftrightarrow x \in \delta\text{-}clA$,

d) $x \in \varphi_{3,4}clA \Leftrightarrow x \in \theta\text{-}clA$.

Theorem 2.7. *Let* $X \in \mathcal{A} \subset P(X)$, $A \subset X$ *and* $\varphi \in O(X, \tau)$. *Then,*

$X \setminus (\mathcal{A}, \varphi)\text{-}intA = (\mathcal{A}, \varphi)\text{-}cl\ (X \setminus A)$.

Proof. Let $x \in X \setminus (\mathcal{A}, \varphi)$-intA. Then $x \notin (\mathcal{A}, \varphi)$-intA. For each $U \in \mathcal{A}(x)$, $\varphi(U) \not\subset A$, i.e. for each $U \in \mathcal{A}(x)$, we have $\varphi(U) \cap (X \setminus A) \neq$. Hence, $x \in (\mathcal{A}, \varphi)$-cl $(X \setminus A)$.

Conversely, let $x \in (\mathcal{A}, \varphi)$-cl$(X \setminus A)$. Then, for each $U \in \mathcal{A}(x)$, we have $(X \setminus A) \cap \varphi(U) \neq$, i.e. for each $U \in \mathcal{A}(x)$, $\varphi(U) \not\subset A$. Hence, $x \notin (\mathcal{A}, \varphi)$-intA.

Theorem 2.8. *Let* $\varphi_1, \varphi_2 \in O(X, \tau)$, $\mathcal{A} = \{\varphi_1(U) : U\ \varphi_1 O(X)\}$. *If* $\mathcal{A} \subset \acute{\mathcal{A}} \subset \varphi_1 O(X)$ *and* $\varphi_2(\varphi_1(U)) \subset \varphi_2(U)$ *for each* $U \in \varphi_1 O(X)$, *then for any* $A \subset X$,

a) $x \in \varphi_{1,2}intA \Leftrightarrow x \in (\acute{\mathcal{A}}, \varphi_2)\text{-}intA$.

b) $x \in \varphi_{1,2}clA \Leftrightarrow x \in (\acute{\mathcal{A}}, \varphi_2)\text{-}clA$.

c) *If* $\varphi_3 \in O(X, \tau)$ *and* $\mathcal{A} \subset \varphi_3 O(X) \subset \varphi_1 O(X)$, *then we have,*

  i) $x \in \varphi_{1,2}intA \Leftrightarrow x \in \varphi_{3,2}intA \Leftrightarrow x \in (\mathcal{A}, \varphi_2)\text{-}intA$.

  ii) $x \in \varphi_{1,2}clA \Leftrightarrow x \in \varphi_{3,2}clA \Leftrightarrow x \in (\mathcal{A}, \varphi_2)\text{-}clA$.

  iii) $A \in \varphi_{1,2}O(X) \Leftrightarrow A \in \varphi_{3,2}O(X)$.

  iv) $A \in \varphi_{1,2}C(X) \Leftrightarrow A \in \varphi_{3,2}C(X)$.

Proof.

a) (:$\Rightarrow$) Let $x \in \varphi_{1,2}intA$. Then there exists an $U \in \varphi_1 O(X, x)$ such that $\varphi_2(U) \subset A$. Since $x \in U \subset \varphi_1(U)$ and $\varphi_1(U) \in \mathcal{A}$, we have $\varphi_1(U) \in \mathcal{A}(x)$ and $\varphi_2(\varphi_1(U)) \subset \varphi_2(\mathrm{U}) \subset A$. Hence, $x \in (\mathcal{A}, \varphi_2)$-intA.

(:$\Leftarrow$) Let $x \in (\mathcal{A}, \varphi_2)$-intA. Then there exists an $U \in \mathcal{A}(x)$ such that $\varphi_2(U) \subset A$. Since $\mathcal{A} \subset \varphi_1 O(X)$, we have $x \in U \in \varphi_1 O(X, x)$ and $\varphi_2(U) \subset A$. Hence, $x \in \varphi_{1,2}$intA.

b) (:$\Rightarrow$) Let $x \in \varphi_{1,2}$clA and $U \in \mathcal{A}(x)$. Since $x \in U$ and $\in \varphi_1 O(X, x)$, we have $\varphi_2(U) \cap A \neq \emptyset$. Hence $x \in (\mathcal{A}, \varphi_2)$-clA.

(:$\Leftarrow$) Let $x \in (\mathcal{A}, \varphi_2)$-clA and $U \in \varphi_1 O(X, x)$. Since $x \in U \subset \varphi_1(U)$. $\varphi_1(U) \in \mathcal{A} \subset \mathcal{A}$, $\varphi_1(U) \in \mathcal{A}$, so, $\varphi_2(\varphi_1(U)) \cap A \neq \emptyset$. Since $\varphi_2(\varphi_1(U)) \subset \varphi_2(U)$, we have $\varphi_2(U) \cap A \neq \emptyset$. Hence, $x \in \varphi_{1,2}$clA.

c) If we take $\varphi_3 O(X)$ or $\mathcal{A}$ instead of $\mathcal{A}$ in *(a)* and *(b)* proofs are clear.

The following important topological spaces will be denoted as follows:

τ_s: the topology of all δ-open sets in (X, τ) (i.e. the semi regularization topology of X with the base RO(X) the family of regular open sets).

τ^α : the topology of all α-open sets in (X, τ).

$\tau^\alpha{}_\theta$: the topology of all θ-open sets in (X, τ^α).

Note 2.9. *The following equailites can be written by using operations from* [5].

For any open set U in the topological space (X, τ) *we have*

$\overline{U} = \theta clU = \tau_s clU = \tau^\alpha clU = pclU$ *and for any* α*-open set U in* (X, τ),

we have $\overline{U} = \tau^\alpha clU = \tau_s clU$.

Hence, for any τ*-closed set K, we have*

$\tau intK = \theta intK = \tau_s intK = preintK = \tau^\alpha intK$.

Example 2.10. *Let* $\varphi_1 = intocl$, $\varphi_2 = cl$, $\varphi_3 = int$, $\varphi_4 = intocloint$.

Then $\varphi_1 O(X) = PO(X) =$ *the family of preopen sets,* $\varphi_3 O(X) = \tau$,

$\varphi_4 O(X) = \tau^\alpha$, $RO(X) \subset \tau_s \subset \tau \subset \tau^\alpha \subset PO(X)$.

$\mathcal{A} = \{\varphi_1(U) : U \in \varphi_1 O(X)\} = \{U^{\underline{\circ}} : U \in PO(X)\} = RO(X) \subset PO(X)$

For each $U \in \varphi_1 O(X) = PO(X)$, $\varphi_2(\varphi_1(U)) = U^{\overline{\underline{\circ}}} \subset \overline{U} = \varphi_2(U)$.

Then by Thm. 1.7 *and Note* 1.8, *we have for any* $A \subset X$,

$x \in \varphi_{1,2}intA \Leftrightarrow x \in \varphi_{3,2}intA \Leftrightarrow$ *there exists an* $U \in \tau(x)$ *such that* $\overline{U} \subset A$

$\Leftrightarrow x \in \theta\ intA$.

And so, $x \in (PO(X), \tau cl)\text{-}intA \Leftrightarrow x \in (\tau, \tau cl)\text{-}intA \Leftrightarrow x \in (\tau, \theta cl)\text{-}intA \Leftrightarrow x \in (\tau, \tau_s cl)\text{-}intA \Leftrightarrow x \in (\tau, \tau_s cl)\text{-}intA \Leftrightarrow x \in (\tau, \tau^\alpha cl)\text{-}intA \Leftrightarrow x \in (\tau, pcl)\text{-}intA \Leftrightarrow x \in (RO(X), \tau cl)\text{-}intA \Leftrightarrow x \in (RO(X), \theta cl)\text{-}intA \Leftrightarrow x \in (RO(X)), \tau_s cl)\text{-}intA \Leftrightarrow x \in (RO(X), \tau^\alpha cl)\text{-}intA \Leftrightarrow x \in (RO(X), pcl)\text{-}intA \Leftrightarrow x \in (\tau_s, \tau cl)\text{-}intA \Leftrightarrow x \in (\tau_s, \theta cl)\text{-}intA \Leftrightarrow x \in (\tau_s, \tau_s cl)\text{-}intA \Leftrightarrow x \in (\tau_s, \tau^\alpha cl)\text{-}intA \Leftrightarrow x \in (\tau_s, pcl)\text{-}intA \Leftrightarrow x \in (\tau^\alpha, \tau cl)\text{-}intA \Leftrightarrow x \in (\tau^\alpha, \tau_s cl)\text{-}intA \Leftrightarrow x \in (\tau^\alpha, \tau^\alpha cl)\text{-}intA$.

$A \in \varphi_{1,2}O(X) \Leftrightarrow A \subset \varphi_{1,2}intA \Leftrightarrow A \subset \theta intA \Leftrightarrow A \in \tau_\theta \Leftrightarrow A \in \tau_\theta^\alpha \Leftrightarrow A \in (\tau_s)_\theta$.

Likewise, the equivalent forms of *(PO(X), τcl)*-clA can be written using Thm. 2.8. and Note 2.9.

The following definitions follow the pattern set in [3].

Definition 2.11. *Let* $\varphi \in O(X, \tau)$, $X \in \mathcal{A} \subset P(X)$ *and* $\mathcal{F}$ *be a filter (or filter-base) in* X *and* $a \in X$. *Then* $\mathcal{F}$ *said to be :*

a) $(\mathcal{A}, \varphi)$*-accumulates to* a *if* $a \in \cap\{(\mathcal{A}, \varphi)\text{-}clF : F \in \mathcal{F}\}$.

b) $(\mathcal{A}, \varphi)$*-converges to* a *if for each* $U \in \mathcal{A}(a)$, *there exists an* $F \in \mathcal{F}$ *such that* $F \subset \varphi_2(U)$.

Definition 2.12. [6] *Let* $\varphi_1, \varphi_2 \in O(X, \tau)$ *and* $\mathcal{F}$ *be a filter (or filterbase) in* X *and* $a \in X$. *Then* $\mathcal{F}$ *said to be :*

a) $\varphi_{1,2}$*-accumulates to* a *if* $a \in \cap\{\varphi_{1,2}clF : F \in \mathcal{F}\}$.

b) $\varphi_{1,2}$*-converges to* a *if for each* $U \in \varphi_1 O(X, a)$, *there exists an* $F \in \mathcal{F}$ *such that* $F \subset \varphi_2(U)$.

Theorem 2.13. *Let* $\varphi \in O(X, \tau)$, $X \in \mathcal{A} \subset P(X)$ *and* $a \in X$.

1) *A filterbase* $\mathcal{F}_b$, $(\mathcal{A}, \varphi)$*-accumulates (*$(\mathcal{A}, \varphi)$*-converges) to* a *iff the filter generated by* $\mathcal{F}_b$, $(\mathcal{A}, \varphi)$*-accumulates (*$(\mathcal{A}, \varphi)$*-converges) to* a.

2) *If* $\mathcal{F}$, $(\mathcal{A}, \varphi)$*-converges to* a, *then* $\mathcal{F}$, $(\mathcal{A}, \varphi)$*-accumulates to* a.

3) *Let* $\mathcal{F} \subset \mathcal{F}'$ *for the filters* $\mathcal{F}$ *and* $\mathcal{F}'$. *Then,*

a) *If* $\mathcal{F}'$, $(\mathcal{A}, \varphi)$*-accumulates to* a, *then* $\mathcal{F}$, $(\mathcal{A}, \varphi)$*-accumulates to* a.

b) *If* $\mathcal{F}$, $(\mathcal{A}, \varphi)$*-converges to* a, *then* $\mathcal{F}'$, $(\mathcal{A}, \varphi)$*-converges to* a.

4) *If* $\mathcal{F}$ *is a maximal filter which* $(\mathcal{A}, \varphi)$*-accumulates to* a, *then* $\mathcal{F}(\mathcal{A}, \varphi)$*-converges to* a.

5) *If φ is regulaer w.r.t. $\mathcal{A}$(i.e. for each $x \in X$ and $U, V \in \mathcal{A}(x)$ there exists an $W \in \mathcal{A}(x)$ such that $\varphi(W) \subset \varphi(U) \cap \varphi(V)$) then, $\mathcal{F}$, $(\mathcal{A}, \varphi)$-accumulates to a iff there exists a filter $\mathcal{F}'$ such that $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{F}'(\mathcal{A}, \varphi)$-converges to a.*

6) *If $\mathcal{A}' \subset \mathcal{A}$ and $\varphi' \geq \varphi$ for the operations $\varphi, \varphi' \in O(X, \tau)$, then $\mathcal{F}$, $(\mathcal{A}', \varphi')$-accumulates ($(\mathcal{A}', \varphi')$-converges) to a whenever $\mathcal{F}$, $(\mathcal{A}, \varphi)$-accumulates ($(\mathcal{A}, \varphi)$-converges) to a.*

Proofs of (4). (5) and (6) can be made in a similar way in [6]. The others are clear.

Example 2.14. *Let $\mathcal{F}$ be a filter in X and $a \in X$.*

a) *If $\varphi_1 = int$, $\varphi_2 = cl$, then*

$\mathcal{F}$ $\varphi_{1,2}$-accumulates to a $\Leftrightarrow$ $\mathcal{F}$ r-accumulates to a,

$\mathcal{F}$ $\varphi_{1,2}$-converges to a $\Leftrightarrow$ $\mathcal{F}$ r-converges to a [2].

b) *If $\varphi_1 = int$, $\varphi_2 = intocl$, then*

$\mathcal{F}$ $\varphi_{1,2}$-accumulates to a $\Leftrightarrow$ $\mathcal{F}$ δ-accumulates to a,

$\mathcal{F}$ $\varphi_{1,2}$-converges to a $\Leftrightarrow$ $\mathcal{F}$ δ-converges to a [3].

3. (A, φ)-COMPACTNESS

Definition 3.1. [6] *Let $\varphi \in O(X, \tau)$, $X \in \mathcal{A} \subset P(X)$ and $A \in P(X)$. Whenever $A \subset \cup\mathcal{U}$ for any subfamily $\mathcal{U}$ of $\mathcal{A}$ if there exists $U_1, ..., U_n \in \mathcal{U}$ such that $A \subset \cup_{i=1}^{n}\varphi(U)$, then we say that the set A is $(\mathcal{A}, \varphi)$-compact set relative to X (for short, $(\mathcal{A}, \varphi)$-compact set).*

We call an $(\mathcal{A}, i)$-compact set as $\mathcal{A}$-compact set.

If X itself $(\mathcal{A}, \varphi)$-compact, then X is called $(\mathcal{A}, \varphi)$-compact space.

For $\varphi_1, \varphi_2 \in O(X, \tau)$, if we take $\mathcal{A} = \varphi_1 O(X)$, we call an $(\mathcal{A}, \varphi_2)$-compact set as $\varphi_{1,2}$-compact set.

Theorem 3.2. [6] *Let $\mathcal{A} = \{\varphi_1(U) : U \in \varphi_1 O(X)\}$ and $\mathcal{A} \subset \mathcal{A}' \subset \varphi_1 O(X)$ and the following statements are given for any $A \in P(X)$.*

a) *A is $\varphi_{1,2}$-compact set.*

b) *A is $(\mathcal{A}', \varphi_2)$-compact set.*

c) *A is $(\mathcal{A}, \varphi_2)$-compact set.*

Then clearly $a \Rightarrow b \Rightarrow c$. If we add the condition $\varphi_2(\varphi_1(U)) \subset \varphi_2(U)$ for each $U \in \varphi_1 O(X)$, then $c \Rightarrow a$.

Theorem 3.3. [6] *Let $X \in \mathcal{A} \subset P(X)$, $\varphi, \varphi' \in O(X, \tau)$. If for each $U \in \mathcal{A}$ $\varphi(U) = \varphi'(U)$, then for any $A \subset X$ the following are equivalent.*

i) *A is $(\mathcal{A}, \varphi)$-compact set.*

ii) *A is $(\mathcal{A}, \varphi')$-compact set.*

Theorem 3.4. *Let $\varphi_1, \varphi_2, \varphi_3 \in O(X, \tau)$, $A \subset X$. If $\varphi_2(U) = \varphi_3(U)$ for each $U \in \varphi_1 O(X)$, then the following are valid.*

1) *For each $A \subset X$,*
    a) $\varphi_{1,2} intA = \varphi_{1,3} intA$.
    b) $\varphi_{1,2} clA = \varphi_{1,3} clA$:
2) $\varphi_{1,2} O(X) = \varphi_{1,3} O(X)$
3) $\varphi_{1,2} C(X) = \varphi_{1,3} C(X)$

Example 3.5. *Let $\varphi_1 = int$, $\varphi_2 = cl$, $\varphi_3 = \tau_s cl$, $\varphi_4 = pcl$, $\varphi_5 = \tau^\alpha cl$, $\varphi_6 = \theta cl$. Then*

$\varphi_{1,2} intA = \varphi_{1,3} intA = \varphi_{1,4} intA = \varphi_{1,5} intA = \varphi_{1,6} intA = \theta intA$.

$\varphi_{1,2} O(X) = \varphi_{1,3} O(X) = \varphi_{1,4} O(X) = \varphi_{1,5} O(X) = \varphi_{1,6} O(X) = \tau_\theta$.

It is well known that a compact space have been characterized in terms of filters. We now establish an analogous results for $(\mathcal{A}, \varphi)$-compactness with the following theorems Thm.3.6. and Thm.3.7. are the adaptation of the Theorem 2.1. in [3].

Theorem 3.6. *Let $\varphi \in O(X, \tau)$ and $X \in \mathcal{A} \subset P(X)$. Then the following are equivalent.*

a) *A is $(\mathcal{A}, \varphi)$-compact set.*
b) *Every filterbase in X which meets A, $(\mathcal{A}, \varphi)$-accumulates to some point in A.*
c) *Every maximal filterbase in X which meets A, $(\mathcal{A}, \varphi)$-converges to some point in A.*
d) *Every filterbase in A, $(\mathcal{A}, \varphi)$-accumulates in X to some point in A.*
e) *Every maximal filterbase in A, $(\mathcal{A}, \varphi)$-converges in X to some point in A.*
f) *For any family W of nonempty sets with $(\cap\{(\mathcal{A}, \varphi)\text{-}clF : F \in W\}) \cap A = \emptyset$, there exists a finite subfamily W' such that $(\cap W') \cap A = \emptyset$.*
g) *For any family W of nonempty sets such that for each finite subfamily W' of W we have $(\cap W') \cap A \neq \emptyset$, then $(\cap\{(\mathcal{A},\varphi)\text{-}clF : F \in W\}) \cap A \neq \emptyset$.*
h) *If $\mathcal{F}_b$ is a filter base such that $(\cap\{(\mathcal{A},\varphi)\text{-}clF : F \in \mathcal{F}\}) \cap A = \emptyset$, then there exists an $F \in \mathcal{F}$ such that $F \cap A = \emptyset$.*

If $\widetilde{\varphi}$ is the dual of φ, then the following statement (i) and (j) are equivalent to each one of the above statements.

i) *For any subfamily Φ of $\mathcal{A}^c$ with $A \cap (\cap \Phi) = \emptyset$, there exists a finite subfamily Φ' of Φ such that $A \cap (\cap\{\widetilde{\varphi}(F) : F \in \Phi'\}) = \emptyset$.*

j) *If* Φ *is a subfamily of* $\mathcal{A}^c$ *such that for each finite subfamily* Φ' *of* Φ *we have* $A\cap(\cap\{\widetilde{\varphi}(F) : F\in\Phi'\}) \neq \emptyset$, *then* $A\cap(\cap\Phi) \neq \emptyset$.

Theorem 3.7. [7] *Let* $\varphi_1, \varphi_2 \in O(X, \tau)$. *Then the following are equivalent.*

a) *A is* $\varphi_{1,2}$*-compact set.*

b) *Every filterbase in X which meets A,* $\varphi_{1,2}$*-accumulates to some point in A.*

c) *Every maximal filterbase in X which meets A* $\varphi_{1,2}$*-converges to some point in A.*

d) *Every filterbase in A,* $\varphi_{1,2}$*-accumulates in X to some point in A.*

e) *Every maximal filterbase in A,* $\varphi_{1,2}$*-converges in X to some point in A.*

f) *For any family W of nonempty sets with* $(\cap\{\varphi_{1,2}clF : F \in W\}) \cap A = \emptyset$, *there exists a finite subfamily* W' *of W such that* $(\cap W') \cap A = \emptyset$.

g) *For any family W of nonempty sets such that for each finite subfamily* W' *of W we have* $(\cap W') \cap A \neq \emptyset$, *then* $(\cap\{\varphi_{1,2}clF : F \in W\})\cap A \neq \emptyset$.

h) *If* $\mathcal{F}_b$ *is a filter base such that* $(\cap\{\varphi_{1,2}clF : F \in \mathcal{F}_b\}) \cap A = \emptyset$ *then, there exists an* $F \in \mathcal{F}_b$ *such that* $F \cap A = \emptyset$.

If $\widetilde{\varphi}_2$ *is the dual of* φ_2, *then the following statement (i) and (j) are equivalent to each one of the above statements.*

i) *For any family* Φ *of* φ_1*-closed sets with* $A\cap(\cap\Phi) = \emptyset$, *there exists a finite subfamily* Φ' *of* Φ *such that* $A\cap(\cap\{\widetilde{\varphi}_2(F) : F \in \Phi'\}) = \emptyset$.

j) *If* Φ *is a family of* φ_1*-closed sets such that for each finite subfamily* Φ' *of* Φ *we have* $A\cap(\cap\{\widetilde{\varphi}_2(F) : F\in\Phi'\}) \neq \emptyset$, *then* $A\cap(\cap\Phi) \neq \emptyset$.

Theorem 3.8. [7] *Let* φ_1 *be monotonous and* $A \subset X$. *If for each U, V* $\in \varphi_1O(X)$, $\varphi_2(U\cup V) = \varphi_2(U) \cup \varphi_2(V)$, *then the following are equivalent.*

a) *A is* $\varphi_{1,2}$*-compact set.*

b) *Each filterbase* $\mathcal{F}_b$ *which is a subfamily of* $\{X\setminus\varphi_2(U) : U \in \varphi_1O(X)\}$ *and meets A,* $\varphi_{1,2}$*-accumulates to some point* $a \in A$.

Example 3.9. *Let* $\varphi_1 = int$ *and* $\varphi_2 = cl$.

Then, $\varphi_1O(X) = \tau$. *For each U, V* $\in \varphi_1O(X)$, $\varphi_2(U\cup V) = \overline{(U \cup V)} = \overline{U} \cup \overline{V} = \varphi_2(U) \cup \varphi_2(V)$. $\{\varphi_2(U) : U \in \varphi_1O(X)\} = \{\overline{U} : U \in \tau\} = RC(X) =$ *the family of regular closed sets in* (X, τ) *and* $\{X\setminus\overline{U} : U_1O(X)\} = RO(X)$.

Now from Thm. 1.22 *and Note.* 1.8 *we can write the following equivalencies.*

X is $\varphi_{1,2}$*-compact* $\Leftrightarrow$ *X is* $(\tau, \tau cl)$*-compact* $\Leftrightarrow$ *X is quasi-H-closed* $\Leftrightarrow$ *each filter-base*

$\mathcal{F}_b \subset RO(X)$ has an r-accumulation point $\Leftrightarrow$ each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau, \theta cl)$-accumulation point $\Leftrightarrow$ each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau, \tau_s cl)$-accumulation point $\Leftrightarrow$ each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau, \tau^{\alpha} cl)$-accumulation point $\Leftrightarrow$ each filterbase $\mathcal{F}_b \subset RO(X)$ has an (τ, pcl)-accumulation point.

Example 3.10. *Let $\varphi_1 = int$, $\varphi_2 = cl$, $\varphi_3 = \theta cl$, $\varphi_4 = \tau_s cl$, $\varphi_5 = \tau^{\alpha} cl$, $\varphi_6 = pcl$. $\varphi_7 = intocloint$, $\varphi_8 = intocl$.*

Then, $\varphi_1 O(X) = \tau$ and for each $U \in \tau$, we have $\varphi_2(U) = \varphi_3(U) = \varphi_4(U) = \varphi_5(U) = \varphi_6(U)$. Dual of φ_2, φ_5, φ_6 respectively $\widetilde{\varphi}_2 = int$, $\widetilde{\varphi}_5 = \tau^{\alpha} int$, $\widetilde{\varphi}_6 = preint$. For each $U \in \tau^{u}$, we have $\varphi_2(U) = \varphi_4(U) = \varphi_5(U)$.

$\varphi_8 O(X) = \{A : A \subset A^{\Omega}\} = PO(X)$.

$\mathcal{A} = \{\varphi_8(U) : U \in PO(X)\} = \{U^{\Omega} : U \in PO(X)\} = RO(X) \subset PO(X) = \varphi_8 O(X)$. $\varphi_2(\varphi_8(U)) = U^{\overline{\Omega}} \subset \overline{U} = \varphi_2(U)$. $\mathcal{A} \subset \tau \subset PO(X)$, $\mathcal{A} \subset \varphi_7 O(X) = \tau^{\alpha} \subset PO(X)$, $\mathcal{A} \subset \tau_s \subset PO(X)$.

In the light of the Example 3.9 and Example 3.10, the following theorem can be written by using Note 2.9, Thm. 3.2. Thm. 3.3 and Thm. 3.6.

Theorem 3.11. *For any topological space (X, τ), the followings are equivalent.*

1) *X is quasi H-closed.*
2) *X is $(\tau, \tau cl)$-compact ($\varphi_{1,2}$-compact for $\varphi_1 = int$, $\varphi_2 = cl$).*
3) *X is $(\tau, \tau_s cl)$-compact ($\varphi_{1,2}$-compact for $\varphi_1 = int$, $\varphi_2 = \tau_s cl$).*
4) *X is $(\tau, \tau^{\alpha} cl)$-compact ($\varphi_{1,2}$-compact for $\varphi_1 = int$, $\varphi_2 = \tau^{\alpha} cl$).*
5) *X is $(\tau, \theta cl)$-compact ($\varphi_{1,2}$-compact for $\varphi_1 = int$, $\varphi_2 = \theta cl$).*
6) *X is (τ, pcl)-compact ($\varphi_{1,2}$-compact for $\varphi_1 = int$, $\varphi_2 = pcl$).*
7) *X is $(\tau^{\alpha}, \tau cl)$-compact ($\varphi_{1,2}$-compact for $\varphi_1 = intocloint$, $\varphi_2 = cl$).*
8) *X is $(\tau^{\alpha}, \tau_s cl)$-compact ($\varphi_{1,2}$-compact for $\varphi_1 = intocloint$, $\varphi_2 = \tau_s cl$).*
9) *X is $(\tau^{\alpha}, \tau^{\alpha} cl)$-compact ($\varphi_{1,2}$-compact for $\varphi_1 = intocloint$, $\varphi_2 = \tau^{\alpha} cl$).*
10) *X is $(PO(X), \tau cl)$-compact ($(\mathcal{A}, \varphi)$-compact for $\mathcal{A} = PO(X)$, $\varphi = \tau cl$).*
11) *X is $(RO(X), \tau cl)$-compact ($(\mathcal{A}, \varphi)$-compact for $\mathcal{A} = RO(X)$, $\varphi = \tau cl$).*
12) *X is $(RO(X), \theta cl)$-compact ($(\mathcal{A}, \varphi)$-compact for $\mathcal{A} = RO(X)$, $\varphi = \theta cl$).*
13) *X is $(RO(X), \tau_s cl)$-compact ($(\mathcal{A}, \varphi)$-compact for $\mathcal{A} = RO(X)$, $\varphi = \tau_s cl$).*
14) *X is $(RO(X), \tau^{\alpha} cl)$-compact ($(\mathcal{A}, \varphi)$-compact for $\mathcal{A} = RO(X)$, $\varphi = \tau^{\alpha} cl$).*
15) *X is $(RO(X), pcl)$-compact ($(\mathcal{A}, \varphi)$-compact for $\mathcal{A} = RO(X)$, $\varphi = pcl$).*

16) *X is* $(\tau_s,\tau cl)$*-compact* $((\mathcal{A},\varphi)$*-compact for* $\mathcal{A}=\tau_s$, $\varphi=\tau cl)$.

17) *X is* $(\tau_s,\tau_s cl)$*-compact* $((\mathcal{A},\varphi)$*-compact for* $\mathcal{A} = \tau_s$, $\varphi=\tau_s cl)$.

18) *X is* $(\tau_s,\theta cl)$*-compact* $((\mathcal{A},\varphi)$*-compact for* $\mathcal{A}=\tau_s$, $\varphi=\theta cl)$.

19) *X is* $(\tau_s, \tau^\alpha cl)$*-compact* $((\mathcal{A},\varphi)$*-compact for* $\mathcal{A} =\tau_s$, $\varphi=\tau^\alpha cl)$.

20) *X is* (τ_s,pcl)*-compact* $((\mathcal{A},\varphi)$*-compact for* $\mathcal{A}=\tau_s$, $\varphi=pcl)$.

Some other equivalent forms of 2 in the above theorem can be written in the following way by using Note 2.9, Example 2.14, Theorem.3.6, Example 3.9.

2a) X is $(\tau, \tau cl)$-compact.

2b) Every filterbase $\mathcal{F}_b$ in X, $(\tau,\tau cl)$-accumulates to some point in X.

2c) Every maximal filterbase $\mathcal{U}_b$ in X, $(\tau,\tau cl)$-converges to some point in X.

2d) For any family W of nonempty sets with $\cap\{\theta clF : F \in W\} = \emptyset$, there exists a finite subfamily W' such that $\cap W' = \emptyset$.

2e) For any family W of nonempty sets such that for each finite subfamily W' of W we have $\cap W' \neq \emptyset$, then $\cap\{\theta clF : F \in W\} \neq \emptyset$.

2f) For any family Φ of τ-closed sets with $\cap\Phi = \emptyset$, there exists a finite subfamily Φ' of Φ such that $\cap\{F^\circ : F \in \Phi'\} = \emptyset$.

2g) If Φ is a family of τ-closed sets such that for each finite subfamily Φ' of Φ we have $\cap\{F^\circ : F \in \Phi'\} \neq \emptyset$, then $\cap\Phi \neq \emptyset$.

2h) For any family Φ of τ-closed sets with $\cap\Phi = \emptyset$, there exists a finite subfamily Φ' of Φ such that $\cap\{\theta intF : F \in \Phi'\} = \emptyset$.

2i) If Φ is a family of τ-closed sets such that for each finite subfamily Φ' of Φ we have $\cap\{\theta intF : F\in\Phi'\} \neq \emptyset$, then $\cap\Phi \neq \emptyset$.

2j) For any family Φ of τ-closed sets with $\cap\Phi = \emptyset$, there exists a finite subfamily Φ' of Φ such that $\cap\{\tau_s intF : F \in \Phi'\} = \emptyset$.

2k) If Φ is a family of τ-closed sets such that for each finite subfamily Φ' of Φ we have $\cap\{\tau_s intF : F \in \Phi'\} \neq \emptyset$, then $\cap\Phi \neq \emptyset$.

2l) Each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau,\tau\text{cl})$-accumulation point.

2m) Each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau,\theta\text{cl})$-accumulation point.

2n) Each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau.\tau_s\text{cl})$-accumulation point.

2o) Each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau,\tau^\alpha\text{cl})$-accumulation point.

2p) Each filterbase $\mathcal{F}_b \subset RO(X)$ has an $(\tau,p\text{cl})$-accumulation point.

In a similar way, some other equivalent forms of each of 3-20 can be written by using Thm.3.6. or \ and Thm.3.8.

Theorem 3.12. [7] *Let* $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$, $(\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq i)$, *and for each* $U \in \varphi_1 O(X)$ *we have* $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$ *then the following are equivalent for any subset* A *of* X.

a) *A is* $\varphi_{1,2}$*-compact set.*

b) *A is* $\mathcal{B}$*-compact set.*

c) *A is* $\varphi_{1,2}O(X)$*-compact set.*

d) *If* W *is any subfamily of* $\{X \setminus \varphi_2(U) : U \in \varphi_1 O(X)\}$ *such that for each finite subfamily* W' *of* W *we have* $A \cap (\cap W') \neq \emptyset$ *then* $A \cap (\cap W) \neq \emptyset$.

e) *If* W *is any subfamily of* $\{X \setminus \varphi_2(U) : U \in \varphi_1 O(X)\}$ *with* $A \cap (\cap W) = \emptyset$ *then there exists a finite subfamily* W' *of* W *such that* $A \cap (\cap W') = \emptyset$.

f) *If* W *is any subfamily of* $\{X \setminus U : U \in \varphi_{1,2}O(X)\}$ *such that for each finite subfamily* W' *of* W *we have* $A \cap (\cap W') \neq \emptyset$ *then* $A \cap (\cap W) \neq \emptyset$.

g) *If* W *is any subfamily of* $\{X \setminus U : U \in \varphi_{1,2}O(X)\}$ *with* $A \cap (\cap W) = \emptyset$ *then there exists a finite subfamily* W' *of* W *such that* $A \cap (\cap W') = \emptyset$.

Theorem 3.13. [7] *Let* $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$. *If* $(\varphi_2 \geq \varphi_1$ *or* $\varphi_2 \geq i)$ *and* $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$ *for each* $U \in \varphi_1 O(X)$, *then the followings are equivalent.*

a) *X is* $\varphi_{1,2}$*-compact.*

b) *X is* $\mathcal{B}$*-compact.*

c) *X is* $\varphi_{1,2}O(X)$*-compact.*

d) *For each* $U \in \varphi_1 O(X)$, $X \setminus \varphi_2(U)$ *is* $\varphi_{1,2}$*-compact set.*

e) *For each* $U \in \varphi_1 O(X)$, $X \setminus \varphi_2(U)$ *is* $\mathcal{B}$*-compact set.*

f) *For each* $U \in \varphi_1 O(X)$, $X \setminus \varphi_2(U)$ *is* $\varphi_{1,2}O(X)$*-compact set.*

g) *For each* $\varphi_{1,2}$*-closed set is* $\varphi_{1,2}$*-compact set.*

h) *For each* $\varphi_{1,2}$*-closed set is* $\mathcal{B}$*-compact set.*

i) *For each* $\varphi_{1,2}$*-closed set is* $\varphi_{,2}O(X)$*-compact set.*

Example 3.14. *Let* $\varphi_1 = int$, $\varphi_2 = intocl$, $\varphi_3 = scl$ *and* $i =$ *identity operation.*

Then $\varphi_1 O(X) = \tau$. *For any* $A \subset X$, $x \in \varphi_{1,2}clA \Leftrightarrow$

$x \in \delta\text{-}clA = \tau_s clA$.

For any $U \in \tau$, $\varphi_2(U) = (\overline{U})^\circ = sclU = \varphi_3(U)$.

$\varphi_{1,2}O(X) = \varphi_{1,3}O(X) = \tau_s$.

X is $\varphi_{1,2}$ *compact space* $\Leftrightarrow$ *X is nearly compact space* $\Leftrightarrow$ *X is* $\varphi_{1,3}$*-compact space.*

For each $U \in \varphi_1 O(X)$, $\varphi_2(U) = (\overline{U})^\circ \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$.

$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = RO(X)$.

For $U \in \varphi_1 O(X) = \tau$, $X \setminus \varphi_2(U) \in RC(X)$, $\widetilde{\varphi}_2 = cloint$, $\widetilde{\varphi}_3 = semi\text{-}int$, $\widetilde{I} = i$.

*Then if we use Thm.*3.3 *and Thm.*3.13 *for these operations, we obtain the following theorem.*

Theorem 3.15. *For any topological space* (X, τ) *the following are equivalent.*

1) *X is nearly compact space.*
2) *X is* $(\tau, intocl)$*-compact space.*
3) *X is* (τ, scl)*-compact space.*
4) *X is* $RO(X)$*-compact space (i.e. X is* $(RO(X), i)$*-compact).*
5) *X is* τ_s*-compact space (i.e. X is* (τ_s, i)*-compact.*
6) *If W is any subfamily of* $RC(X)$ *such that for each finite subfamily* W' *of W we have* $\cap W' \neq \emptyset$ *then* $\cap W \neq \emptyset$.
7) *If W is any subfamily of* $RC(X)$ *with* $\cap W = \emptyset$ *then there exists a finite subfamily* W' *of W such that* $\cap W' = \emptyset$.
8) *If W is any family of* τ_s*-closed sets such that for each finite subfamily* W' *of W we have* $\cap W' \neq \emptyset$ *then* $\cap W \neq \emptyset$.
9) *If W is any family of* τ_s*-closed sets with* $\cap W = \emptyset$ *then there exists a finite subfamily* W' *of W such that* $\cap W' = \emptyset$.
10) *Each regular closed set is* $(\tau\text{-}intocl)$*. compact set.*
11) *Each regular closed set is* (τ, scl)*-compact set.*
12) *Each regular closed set is* $RO(X)$*-compact set.*
13) *Each regular closed set is* τ_s*-compact set.*
14) *Each* τ_s*-closed set is* $(\tau\text{-}intocl)$*-compact set.*
15) *Each* τ_s*-closed set is* $RO(X)$*-compact set.*
16) *Each* τ_s*-closed set is* τ_s*-compact set.*

Some other equivalent forms of 2 *in the above Thm. can be written in the following way by using Thm.*3.6.

2a) *Every filterbase* $\mathcal{F}_b$ *in X,* δ*-accumulates to some point in X.*

2b) *Every maximal filterbase* $\mathcal{U}_b$ *in* X, *δ-converges to some point in* X.

2c) *For any family* W *of nonempty sets with* $\cap\{\delta clF : F \in W\} = \emptyset$, *there exists a finite subfamily* W' *of* W *such that* $\cap W' = \emptyset$.

2d) *For any family* W *of nonempty sets such that for each finite subfamily* W' *of* W *we have* $\cap W' \neq \emptyset$, *then* $\cap\{\delta clF : F \in W\} \neq \emptyset$.

2e) *For any family* Φ *of* τ*-closed sets with* $\cap\Phi = \emptyset$, *there exists a finite subfamily* Φ' *of* Φ *such that* $\cap\{F^{\bar{\circ}} : F \in \Phi'\} = \emptyset$.

2f) *If* Φ *is a family of* τ*-closed sets such that for each finite subfamily* Φ' *of* Φ *we have* $\cap\{F^{\bar{\circ}} : F \in \Phi'\} \neq \emptyset$, *then* $\cap\Phi \neq \emptyset$.

In a similar way some other equivalent forms of each of 3-12 can be written by using Thm.3.6.

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Jour. Pure Math., Vol. 23, 2006, pp. 15-36

SOME ASPECTS OF SEMIRING GROUPS WITH A.C.C. ON ANNIHILATORS

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ABSTRACT : In this paper we deal with hemirings and H-hemiring groups, together with finiteness condition on annihilators of such algebraic structures. To be more specific we deal with an Goldie H-hemiring group E that is if

1) E has no infinite independent family of non-zero H-ideals. And

2) Annihilators in H of subsets of E satisfy the ascending chain condition (a.c.c.) (under set inclusion).

The results proved here may be considered analogous to those of some Notherian module in some special cases. The results proved here finally lead to the fact that the collection of associated operating sets of E is finite and there exists a so-called primary decomposition of zero of such a Goldie H-hemiring group.

2005 Mathematical subject classification (Amer.Math.Soc): 16 Y 60, 20 M 25, 20 M 14., 20M 12.

Key words and pharases : hemiring, halfring, H-sub hemigroup, H-ideal, H-hemiring group, H-sub hemiring group, Goldie H-hemiring group, Essential H-sub hemi group, independent family of H-sub hemi group, essential as H-ideal, Quasisubtractive hemiring, annihilator of subset, prime ideal.

1. INTRODUCTION

It is well known that the set of natural number is the trivial semiring structure which can be derived from fundamentals coined by G-Peano, and this N, the obvious semiring, can be made into a ring defined by an equivalence relation on **N X N** in which the each equivalence class (1,1),(1,2) and (2,1) etc. respectively represents the integer zero, the positive integer 1 and the negative integer -1 etc. For the sake of beauty of its abstractness, we would like to exhibit the possible geometrical link between the semiring N of natural numbers and that of the ring **Z** of integers with a hope to get some indications for future development.

A non-empty set S is a **Semiring** if the set has two associative binary operations defined on it which are called addition and multiplication and which satisfies the following properties:

1) The multiplication satisfies both the distributive laws together with the addition.
2) Addition is commutative.

and this triple is a **hemiring** if S contains an element '0' such that $x + 0 = x$ and $x.0 = 0 = 0.x$, for all $x \in S$. Element '0' is called a 'zero' of S. It can be seen that if such a '0' exists in a semiring then it is unique.

S is said to have unity '1' if $1.s = s = s.1$, for all $s \in S$.

A semiring may have a unity or mayn't also.

S is **halfring** if it satisfies the additive law of cancellation, namely, $x + a = x + b \Rightarrow a = b$, for all $x, a, b \in S$.

A non-empty subset A of a semiring (S, +,.) is a **sub-semiring** of S if the corresponding triple (T, +, .) is also a semiring.

For the sake obvious reason, we like to present the construction of the ring of integers from the set of natural numbers[7]. Theoritically a ring can be constructed form a given halfring as follows:

In case of a halfring (H, +,.), we define a relation ~ in H×H such that $(a, b) \sim (c, d)$ if $a + b = c + b$. and this is an equivalence relation. So it partitions $H \times H$ into mutually disjoint equivalence classes.

The equivalence class equivalent to (a, b) is denoted by a-b that is,

$(a - b) = \{(x, y) \in H \times H$ such that $(x, y) \sim (a, b)\}$. And the family of these classes will be denoted by $\overline{H}$. Now, in $\overline{H}$ we define '+' and '.' as follows

1) $(a - b) + (c - d) = (a + c) - (b + b)$, 2) $(a - b).(c - d) = (ac + bd) - (ad + bc)$.

The well defineness of theses operations can be checked, finally giving the ring structure of $\overline{H}$.

An H-hemiring group E over H (or $_HE$) is such that

1) (E, +) is commutative hemi group, 2) H, +. .) is a hemiring

there is a map $H \times E \rightarrow E$ $(h, e) \rightarrow he$ such that

$(h_1 + h_2)e = h_1e + h_2e$; $h_1(e_1 + e_2) = h_1e_1 + h_1e_2$; $(h_1h_2)e = h_1(h_2e)$; $h_1, h_2 \in H$; $e_1, e_2 \in E$. It can easily be seen that every hemiring H is a left as well as a right H-hemiring group over itself.

A non-empty subset A of an H-hemiring group E over H is a **sub hemiring group** of E if the subset A is also an H-hemiring group.

Left H-sub hemigroup of a hemiring H is a non-empty subset A of H such that $a+b \in A$, for every $a, b \in A$ and such that $ha \in A$. for every $h \in H$, $a \in A$.

Similarly right H-sub hemigroup and two-sided H-sub hemigroup may be defined.

An **H-sub hemiring group** of an H-hemiring group E is a non-empty subset A of E such that $x + y \in A$ for every $x, y \in A$, and such that $hx \in A$ for every $h \in H$ and $x \in A$.

If E = H then H-sub hemiring group of E will be left H-sub hemigroup of H.

A left (right, two-sided) H-sub hemigroup A of a hemiring H is a **left (right, two-sided) H-ideal** if, whenever $x, y \in H$ such that x and $x + y \in A$, then $y \in A$.

Let $A \subseteq E$. Then A is an **H-ideal** of E if, whenever $x, y \in E$ such that $x \in A$ and $x + y \in A$, then $y \in A$.

If E = H then H-ideal of E will be **left H-ideal** of H.

A left (right, two-sided) H-sub hemigroup of a hemiring H is **generated by a non-empty subset S** if it is the intersection of all left (right, two-sided) H-sub hemigroups of H. In other word it is the smallest left (right, two-sided) H-sub hemigroup containing S[7].

If $S = \{x\}$ then the H-sub hemigroup generated by x will be denoted by $<x>$

If A is a left (right, two-sided) H-sub hemigroup of H then [A] will denote the smallest left (right, two-sided) H-ideal containing A or the left (right, two-sided) H-ideal generated by A. If A is a left H-sub hemigroup of H then

$$[A] = \{x : a_1 + x = a_2 \text{ for some } a_1, a_2 \in A\}.$$

For an H-ideal A of a hemiring H, the set $x + A = \{y \mid x + a = y + a' \text{ for some } a, a' \in A\}$ is a coset of A. An H-sub hemiring group A of E is an **essential H-sub hemigroup** of E if for each non-zero H-sub hemiring group K of E we have $A \cap K \neq 0$ (equivalently for an H-sub hemiring group K of E, $A \cap K = 0 \Rightarrow A = 0$) and is denoted by $A \leq_e E$.

An **H-ideal A of E is essential** if for any non-zero H-ideal K of E we have $A \cap K \neq 0$ (equivalently for an H-ideal K of E, $A \cap K = 0 \Rightarrow A = 0$) and is denoted by $A \leq_e E$.

An ordered family $\{E_1, E_2, \ldots E_n\}$ of H-sub hemiring groups (H-ideals) of E is an **independent family** if $\{E_1 + E_2 + \ldots + \hat{E}_t + \ldots E_n\} \cap E_t = 0$, for $1 \leq t \leq n$

Where $\wedge$ stands for deletion of the symbol underneath.

An infinite family $\{E_1, E_2, \ldots\}$ is **independent** if for any $n \in N$ the corresponding finite family $\{E_1, \ldots, E_n\}$ is independent.

An H-sub hemiring group (H-ideal)A of E is **closed H-sub hemiring group (closed H-ideal)** of E if A has no proper essential extension of E. (equivalently if A and K are H-sub hemiring groups of E then $A \leq {}_e K < E \Rightarrow A = K$ and is denoted by $A \leq_e E$).

An **annihilator of subset A** of a H-hemiring group E over H is the collection of those elements of H which annihilates each element of A and is denoted by $Ann_H(A)$ (or A(A)), thus

$$A(A) = \{h \in H | ha = 0, \text{ for all } a \in A\}.$$

If E = H then Ann(A) is written as $l_H(A)$ or in short l(A) and it is left annihilator of A in H.

If I is a sub-hemiring of H and $H_1 \subseteq H$ then left annihilator of H_1 in I is defined as

$l_I(H_1) = \{x \in I | xH_1 = (O)\}$ and right annihilator of H_1 in I is defined as $r_I(H_1) = \{x \in I | H_1 x = (0)\}$

If I = H then $l_I(H_1) = l_H(H_1)$ and $r_I(H_1) = r_H(H_1)$ are written as $l(H_1)$ and $r(H_1)$ respectfully.

Consider an **H-hemiring group** E over H. E is **Goldie H-hemiring group** if

(i) E has no infinite independent family of H-ideals and

(ii) the a.c.c. is satisfied for the annihilators of H-sub hemiring group of E in H.

The homomorphic image of a Goldie H-hemiring group need not be a Goldie H-hemiring group. For in case of a Goldie ring, a homomorphic image of a Goldie ring need not be a Goldie ring [1]. A Goldie H-hemiring group is fully Goldie if every homomorphic image of it is Goldie H-hemiring group.

A non-zero H-sub hemiring group (H-ideal) P of E is a **prime H-sub hemiring group** of E if every non-zero H-sub hemiring group Q of P, A(P) = A(Q).

If for each H-sub hemiring group P of the H-hemiring group E, A(P) = A(E), then E is called a **prime H-hemiring group.**

The collection $\mathcal{A}(E) = \{A(E') |$ for some prime H-sub hemigroup E' of E$\}$, is the family of associated primes of E.

If $\mathcal{A}(E)$ is singleton then H-hemiring group E is called **H-primary.**

Let E be an H-hemiring group with closed H-ideals $E_1, E_2, \dots E_l$ satisfying the following conditions:

1) The closed H-ideals $E_1, E_2, \dots, E_l$ have Zero intersection, but

$E_1 \cap E_2 \cap \dots \cap \hat{E}_i \cap \dots \cap E_l \neq (0)$, for any i = 1, ... t (The symbol ^ denotes omission of E_i).

2) Each quotient H-hemiring group E/E_i is a primary H-hemiring group such that

$\mathcal{A}(E/E_i) \neq \mathcal{A}(E/E_j)$ for $i \neq j$ [2]

Then the expression $(0) = E_1 \cap E_2 \cap \ldots \cap E_t$ is an **H-Primary decomposition** of 0 in E.

We prove in this section the existence of an H-primary decomposition of 0 in some special Goldie H-hemiring groups.

Note. In an H group E, the subset

$\mathbf{Z_1(E)} = \{u \in E | Lu = (0)$ for some essential H-sub hemiring group L of ${}_H H\}$ plays an important role in what follows

A hemiring H is a **Quasi-subtractive hemiring** if for H-ideal I and J; $h \in H$, $i \in I$, $j \in J$ and $j = h + i$ then there exists $j' \in J$ if and only if $0 = h + i + j'$.

Let $\{A_i\}$ be a family of H-ideals. Then the sum $(\sum_i A_i)$ is ***essential as an H-ideal*** if for any H-ideal B of H, we have $(\Sigma A_i) \cap B = 0 \Rightarrow B = 0$.

Let $\{I_\alpha : \alpha \in A\}$ be a collection of non-zero left H-ideals of a hemiring H. $I = \Sigma I_\alpha$ is said to be **direct** if every element of I can be uniquely expressed in the form $\sum a_{\alpha_i}$ where $a_{\alpha_i} \in I_{\alpha_i}$.

Let $\{E_i\}$ be any family of H-hemiring groups then $\{(\ldots, e_i \ldots)\} | e_i \in E_i$, $e_i = 0$ for all but a finite number of $i\}$ is defined as the **direct sum** of E_i's and is denoted by E_i.

Each $\oplus E_i$ is a **direct summand** of $\oplus E_i$.

Also if $\{H_i\}$ is a family of hemirings then

$\{(\ldots, h_i, \ldots) | h_i \in H_i$, $h_i = 0$ for all but a finite number of $i\}$ is defined as the direct sum of $H_i^{'}$ and is denoted by $\oplus H_i$. Each H_i is a direct summand of $\oplus H_i$.

Lemmas. (i) Direct sum $\bigoplus_{i=1}^{t} E_i$ of H-hemiring groups is an H-hemiring group if the elements of $\bigoplus_{i=1}^{t} E_i$ are added co-ordinate wise and for $h \in H$, $e_i \in E_i$, $h(\ldots, e_i, \ldots) = (\ldots, he_i, \ldots)$, $i = 1, 2, \ldots t$.

(iii) Direct sum $\bigoplus_{i=1}^{t} H_i$ of hemirings is a hemiring if the elements of $\bigoplus_{i=1}^{t} H_i$ are added co-ordinate wise and the multiplication defined by $(a_1, a_2, \ldots a_t)(b_1, \ldots b_t) = (a_1 b_1, a_2 b_2, \ldots a_t b_t)$, for $a_i, b_i \in H_i$, $i = 1, 2, \ldots t$.

A sum $A = \sum_{i=1}^{t} A_i$ for a finite number of H-sub hemigroups (H-ideals) A_i of a hemiring

H is a direct sum of H-sub hemigroups (H-ideals) provided each element of A is uniquely expressible in the form $\sum_{i=1}^{t} a_i$ where $a_i \in A_i$, i = 1, 2, ...t and this direct sum is denoted by $A = A_1 \oplus A_2 \oplus \dots \oplus A_t$.

Suppose (S, +, .) and (T, +, .) are two semirings. A mapping of S → T is a semiring homomorphism if for all $x, y \in S$, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.

If f: S → T is a one-one-semiring homomorphism then we say S is embedded in T as a sub semiring.

We define a map $\phi : H \to \overline{H}$ such that $H_h \to h{-}0$. $\phi(h) = h{-}0$, for all $h \in H$

It can be easily seen that H is a hemiring monomorphism. Hence H is embedded into $\overline{H}$.

If E and E_1 are two H-hemiring groups over the hemiring H then a mapping f: $E \to E_1$ is a **H-homomorphism** if for all $x, y \in E$, $h \in H$, $f(x + y) = f(x) + f(y)$ and $f(hx) = hf(x)$.

Kernel of H-homomorphism is defined as $(\ker f) = \{x \in E | f(x) = 0\}$.

If ker f = (0) then f is a H-monomorphism and E is embedded in E_1.

When E is embedded in E_1 then we consider E as an

H-sub hemiring group of E_1. An H-homomorphism f is an H-epimorphism if f is surjective. If H-homomorphism f is injective as well as surjective then f is an H-isomorphism. It is denoted by $E \cong E_1$.

Here we prove:

1) Let E is a Goldie H-hemiring group over hemiring H. If the set $A(E)$ is union of two disjoint sets X and Y, then there exists a closed H-ideal E′ such that

   $\mathcal{A}(E') = X$ and $\mathcal{A}(E/E') = Y$.

2) Let E is a Goldie H-hemiring group as above. Then $A(E)$ is finite.

3) Let the H-hemiring group E be fully Goldie as above

   (i) There exists an H-primary decomposition of 0 in E.

   (ii) If $E_1, E_2, \dots, E_t$ are H-ideals and $E_1 \cap E_2 \cap \dots \cap E_t$ is an H-primary decomposition of 0 in E then

   $\mathcal{A}(E) = A(E/E_1) \cup \mathcal{A}(E/E_2) \cup \dots \cup \mathcal{A}(E/E_t)$.

4) E is a Quasi-subtractive Goldie H-hemiring group with $Z_l(E) = 0$ as above and such that an essential H-sub hemigroup of E is essential as an H-ideal also. Then the Annihilators of subsets of E Satisfies the descending chain condition.

2. PRELIMINARIES

2.1. SECTION.

2.1.1. Proposition. Let A be an H-ideal of a halfring H and let

$\overline{A} = \{a - a' \in \overline{H}: a, a' \in A\}$. $\overline{A}$ is an H-sub hemigroup of $\overline{H}$.

Proof. $A \neq \phi$. Implies that $\overline{A} \neq \phi$.

If $a - a'$ and $a_1 - a_1' \in \overline{A}$ then $(a - a') - (a_1 - a_1') = (a + a_1') - (a' + a_1) \in \overline{A}$. Moreover

$(h - h') \in \overline{H}$ then $(h - h')(a - a') = (ha + h'a') - (ha' + h'a) \in \overline{A}$.

Similarly $(a - a')(h - h') \in \overline{A}$.

Let A be a H-ideal of a hemiring. For $x \in H$ let

$x + A = \{y : x + a = y + a' \text{ for some } a, a' \in A\}$. Then $x + A$ is the coset of A.

2.1.2. Proposition. Let $x + A$ and $y + A$ be two cosets of A. Then either $x + A = y + A$ or $x + A \cap y + A = \phi$.

Proof. Suppose that $x + A \cap y + A \neq \phi$.

Let $g \in x + A \cap y + A$ Then $g + a_1 = x + a_1'$ and $g + a_2 = y + a_2'$. If $t \in x + A$, $t - a = x + a'$

Thus $t + a + a_1' + a_2 = x + a' + a_1' + a_2$

$= g + a_2 + a' + a_1$

$= y + a_2' + a_1 + a'$ and $t \in y + A$. so $x + A \subseteq y + A$.

Similarly $y + A \subseteq x + A$.

The above lemma guarantees that the set of cosets of A is a partition of H. It is routine to verify the following theorem.

In 1958 A.W Goldie [3] showed that prime ring with a.c.c. on left annihilators and not containing infinite direct sums of left ideals possesses a left quotient ring. Goldie [4] later extended these results to semi-prime rings.

2.1.3. Proposition. Let A be an H-ideal, then the set of cosets $\{x + A\}$ together with the operations.

$(x + A) + (y + A) = x + y + A$ and $(x + A).(y + a) = xy + A$ is a hemiring.

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The hemiring defined in above theorem is the **quotient hemiring** of H with respect to A and is denoted by H/A.

2.1.4. Proposition. Let H be a halfring and let A be an H-ideal of H. Then H/A is a halfring.

Proof. By theorem 2.1.3, H/A is a hemiring.

If $(x + A) + (y + A) = (x + A) + (z + A)$, then $x + y + A = x + z + A$.

So $x + y + a = x + z + a'$ for some $a, a' \in A$. Therefore $y + a = z + a'$ and $y \in z + A$. Thus by lemma 2.1.2, $y + A = z + A$. Therefore H/A is a halfring.

2.1.5. Theorem. If A is an H-ideal of a halfring H then $(\overline{H / A}) \cong \overline{H}/\overline{A}$.

Proof. Define a mapping by $\pi(\overline{H / A}) \to \overline{H}/\overline{A}$ by $[(h + a) - (h_1 + A)] = h - h_1 + \overline{A}$.

If $(h + A) - (h_1 + A) = (k + A) - (k_1 + A)$ then $h + k_1 + A = h_1 + k + A$. Thus $h + k_1 + a = h_1 + k + a'$ for some $a, a' \in A$.

Therefore $(h - h_1) = (k - k_1) + (a' - a)$ and $(h - h_1) + A = (k - k_1) + \overline{A}$. Hence π is a well-defined function. π is a homomorphism since

$[(h + k + a) - (h_1 + k_1 + a)]\pi = (h + k) - (h_1 + k_1) + \overline{A}$

$= (h - h_1 + \overline{A}) + (k - k_1 + \overline{A})$

$[(h + A) - (h_1 + A)]\pi + [(k + A) - (k_1 + A)]\pi$ and

$[(h + A) - (h_1 + A)].[(k + A) - (k_1 + A)]\pi$

$= [(hk + h_1k_1 + A) - (hk_1 + h_1k + A)]\pi$

$= (hk + h_1k_1) - (hk_1 + h_1k) + \overline{A}$

$= [(h - h_1) + \overline{A}].[(k - k_1) + \overline{A}]$

$= [(h - h_1) + \overline{A}].[(k - k_1) + \overline{A}]$

$= [(h + A) - (h_1 + A)]\pi.[(k + A) - (k_1 + A)]\pi$

Clearly π is onto $\overline{H}/\overline{A}$. Finally π is 1-1. For if $(h + A + h_1 + A)$ is in the kernel of π, $h - h_1 \in \overline{A}$. Thus $h - h_1 = a - a_1$, for some $a, a_1 \in A$. Hence $h + a_1 = h_1 + a$ or $h + A = h_1 + A$.

2.16. Lemma. If $A \subseteq E$ then Ann(A) is an H-ideal of E.

Proof. x and $x + y \in$ Ann(A)

Therefore, $xa = 0$ and

$(x + y)a = 0$ for all $a \in A$.

$\Rightarrow xa + ya = 0$.

$\Rightarrow ya = 0 \Rightarrow y \in \mathrm{Ann}(A)$.

Therefore, Ann(A) is an H-ideal of E.

2.1.7. Cor. If $E = H$ then $\mathrm{Ann}(A) = l(A)$ is a left H-ideal of H.

If $a \in H$ then $l(\{a\})$ is written as $l(a)$ and it is a left H-ideal of H.

2.1.8. Lemma. If A is an H-sub hemigroup of E then Ann(A) is an H-sub hemigroup of E.

Proof. Ann(A) is a H-ideal of E.

Let $x, y \in \mathrm{Ann}(A)$.

$\Rightarrow xa = 0$ and $ya = 0$.

$\Rightarrow xa + ya = 0$

$\Rightarrow (x + y)a = 0 \Rightarrow x + y \in \mathrm{Ann}(A)$.

If $x \in \mathrm{Ann}(A)$ and $h \in H$ then $(xh)A = x(hA) \subseteq Xa = (0) \Rightarrow xh \in \mathrm{Ann}(A)$.

Therefore Ann(A) is an H-sub hemigroup of E.

A non-empty subset H_1 of a hemiring H is

i) a right H-subset of H if $H_1.H \subseteq H_1$.

ii) a left H-subset of H if $H.H_1 \subseteq H_1$.

iii) an invariant subset of H if $H.H_1 \subseteq H_1$ and $H_1.H \subseteq H_1$.

2.1.9. Lemma. If A is a left H-subset of H then $l(A)$ is a two-sided H-sub hemigroup of H.

Proof. $l(A)$ is a left H-sub hemigroup of H.

Now, if $x \in l(A)$ and $h \in H$ then $(xh)A = x(hA) \subseteq xA = (0) \Rightarrow xh \in l(A)$.

Hence $l(A)$ is a right H-sub hemigroup of H.

Therefore $l(A)$ is a two-sided H-sub hemigroup of H.

2.1.10. Corollary. If $A \subseteq H$ then $l(l(A))$ is a two-sided H-sub hemigroup of H.

2.1.11. Lemma. If $A \subseteq H$ then $r(A)$ is a right H-subset of H.

Proof. Let $x \in r(A)$ and $h \in H$ then

$A(xh) = (Ax)h = 0$

$\Rightarrow xh \in r(A)$, for all $h \in H$.

Therefore $r(A)$ is a right H-subset of H. It is noted that if $a \in H$ then $r(\{a\})$ is written as $r(a)$ and it is a right H-subset of H.

2.1.12. Lemma. If A is a right H-subset of H then $r(A)$ is an invariant subset of H

Proof. We have $r(A)$ is a right H-subset of H.

Now, $x \in r(A)$ and $h \in H$ then $A(hx) = (Ah)\, x \subseteq Ax = (0)$.

$\Rightarrow hx \in r(A)$, for all $h \in H$.

Hence $r(A)$ is a left H-subset of H.

Therefore $r(A)$ is an invariant subset of H.

2.1.13. Cor. If $A \subseteq H$ then $r(r(A))$ is an invariant subset of H.

2.1.14. Lemma. If $S \subseteq H$ and $A \subset E$ then (i) $Sr_E(S) = (0)$ and $Ann(A)A = (0)$.

2.1.15. Corollary. If $T \subseteq S \subseteq H$ and $P \subseteq A \subseteq E$ then $Tr_E(S) = (0)$ and $Ann(A)P = (0)$.

2.1.16. Lemma. If $A \subseteq E$, $B \subseteq E$ such that $A \subseteq B$ then $Ann(B) \subseteq Ann(A)$.

2.1.17. Corollary. If $E = H$ then $l(B) \subseteq l(A)$ when $A \subseteq B$.

2.1.18. Lemma. If $P, Q \subseteq H$ such that $P \subseteq Q$ then $r_E(Q) \subseteq r_E(P)$.

2.1.19. Corollary. If $E = H$ then $r(Q) \subseteq r(P)$ when $P \subseteq Q$.

2.1.20. Lemma. If $S \subseteq H$ and $A \subseteq E$ then

a) $r_E(Ann(r_E(S))) = r_E(S)$ and b) $Ann(r_E(Ann(A))) = Ann(A)$.

Proof. we have $S\, r_E(S) = (0) \Rightarrow S \subseteq Ann(r_E(S))$

$\Rightarrow r_E(Ann(r_E(S))) \subseteq r_E(S)$

Also we have $Ann(A)A = 0 \Rightarrow A \subseteq r_E(Ann(A))$

replacing A by $r_E(S)$ we have $r_E(S) \subseteq r_E(Ann(r_E(S)))$

Therefore $r_E(S) = r_E(Ann(r_E(S)))$

Again we have $Sr_E(S) = (0) \Rightarrow S \subseteq Ann(r_E(S))$

replacing S by $Ann(A)$

we have $Ann(A) \subseteq Ann(r_E(Ann(A)))$

Also we have $\mathrm{Ann}(A)A = 0$

$\Rightarrow A \subseteq r_E(\mathrm{Ann}(A))$

$\Rightarrow \mathrm{Ann}(A) \supseteq \mathrm{Ann}(r_E(\mathrm{Ann}(A)))$

Therefore $\mathrm{Ann}(A) = \mathrm{Ann}(r_E(\mathrm{Ann}(A)))$.

2.1.21. Corollary. If $E = H$ then we get

$r(l(r(S))) = r(S)$ and $l(r(l(S))) = l(S)$.

2.1.22. Lemma. Let A be an H-sub hemigroup of H and $S \subseteq A$.

then $r_H(S) \cap A = r_A(S)$ and $l_H(S) \cap A = l_A(S)$.

Proof. Let $x \in r_A(S)$ Then $Sx = (0)$ and $x \in A$ $(\subseteq H)$

$\Rightarrow x \in r_H(S) \cap A$.

$\Rightarrow r_A(S) \subseteq r_H(S) \cap A$.

Conversely, Let $y \in r_H(S) \cap A$

$\Rightarrow y \in r_H(S)$ and $y \in A$

$\Rightarrow Sy = (0)$ and $y \in A(\subseteq H)$

$\Rightarrow y \in r_A(S)$

$\Rightarrow r_H(S) \cap A \subseteq r_A(S)$.

Therefore $r_H(S) \cap A = r_A(S)$. Similarly we can show that $l_H(S) \cap A = l_A(S)$.

2.1.23. Lemma. If $x \in H$ such that $r(x^t) = r(x^{t+1})$ for some $t \geq 1$ then $r(x^{t+m}) = r(x^t)$ for all positive integer m.

Proof. If $m = 1$ then $l(x^{t+1}) = l(x^t)$ which is our hypothesis. Thus the given result is true from $m = 1$

Now, let the result is true for $m = r > 1$. i.e. $l(x^{t+r}) = l(x^t)$ ..... (i)

Let $y \in l(x^{t+r+1})$ then $yx^{t+r+1} = 0$.

$\Rightarrow (yx)x^{t+r} = 0 \Rightarrow yx \in l(x^{t+r}) = l(x^t)$, by (i)

$\Rightarrow yx \in l(x^t) \Rightarrow yx.x^t = 0 \Rightarrow yx^{t+1} = 0 \Rightarrow y \in l(x^{t+1}) = l(x^t)$, by hypothesis.

Therefore $l(x^{t+r+1}) \subseteq l(x^t)$. But since $r + 1$ is positive integer, then clearly $l(x^t) \subseteq l(x^{t+r+1})$.

Thus we have $l(x^{t+r+1}) = l(x^t)$.

Therefore, when the result is true for $m = r$ then it is true for $m = r + 1$. Hence by induction we have $l(x^{t+m}) = l(x^t)$ for all positive integer m.

Similarly we get the following

2.1.24. Lemma. If $x \in H$ such that $r(x^t) = r(x^{t+1})$ for some $t \geq 1$ then $r(x^{t+m}) = r(x^t)$ for all positive integer m.

2.2. SECTION.

2.2.1. Lemma. If a Quasi-subtractive H-hemiring group E has no infinite direct sum of H-ideals then E satisfies the a.c.c. on closed H-ideals of E.

Proof. Suppose E does not satisfy the a.c.c. on closed H-ideals. Then there exists a strictly ascending chain $E_1 < E_2 < \ldots$ of closed H-ideals of E.

Since $E_n \leq_c E$, we have $E_n \not<_e E_{n+1}$

Therefore E_{n+1} must have a non-zero H-ideal (say) C_n such that $E_n \cap C_n = 0$,

And $C_n \subseteq E_{n+1}$. This is true for each n. We claim for any $t \in Z^+$ $i \leq t$.

then $(C_1 + C_2 + \ldots + \hat{C}_i + \ldots + C_t) \cap C_i = 0$.

here, $(C_1 + C_2 + \ldots + \hat{C}_i + \ldots + C_t) \cap C_i \subseteq (E_2 + \ldots + E_i + C_{i+1} \ldots + C_t) \cap C_i$.

Suppose $c_i = e_2 + e_3 + \ldots + e_i + c_{i+1} + \ldots c_t$ is an element of

$(E_2 + \ldots + E_i + C_{i+1} \ldots + C_t) \cap C_i$ then

$c_n \in C_n \cap (E_2 + \ldots + E_i + C_{i+1} + \ldots + C_n)$

$c_n \in C_n \cap (E_2 + \ldots + E_{i+1} + \ldots + E_n)$. $(c_n \subseteq En + 1)$

$\subseteq E_n \cap C_n = 0$.

Therefore $c_n = 0$. for all n; similarly, $c_{n-1} = 0, \ldots c_i = 0$

So $(E_2 + \ldots + E_i + C_{i+1} \ldots + C_t) \cap C_i = 0$.

So $(C_1 + C_2 + \ldots + \hat{C}_i + \ldots + C_t) \cap C_i = 0$.

Hence $\{C_1, C_2, \ldots\}$ is an independent family of non-zero H-ideals of E. Since E has no infinite independent family of non-zero H-ideals of E or E has no infinite direct sum of H-ideals we can not have a strictly ascending chain on closed H-ideals of E. Thus E satisfies the a.c.c. on closed H-ideals of E.

2.2.2. Lemma. Let E be H-hemiring group over the hemiring H. E satisfies the a.c.c. for its annihilators of E in H. Then $\mathcal{A}(E) \neq \phi$ iff $E = 0$.

Proof. Assume $E = 0$.

Then E has no prime H-sub hemiring group.

Hence $\mathcal{A}(E) = \phi$.

Again if $E \neq 0$, Consider $\Im = \{A(E^*) | E^*$ is an H-sub hemiring group of E$\}$.

Since E is goldie, $\Im$ has a maximal element say A(M).

Now let M' ($\neq 0$) be an H-sub hemiring group E such that $M' \subseteq M$.

Then $A(M') \supseteq A(M) \Rightarrow A(M') = A(M)$ (since A(M) is maximal).

Therefore M is a prime H-sub hemiring group of E.

Hence $A(M) \in \mathcal{A}(E)$, that is $\mathcal{A}(E) \neq \phi$.

2.2.3. Lemma. Let E be H-hemiring group with an exact sequence

$0 \to E' \xrightarrow{g} E \xrightarrow{f} E'' \to 0.$

Then $\mathcal{A}(E') \subseteq \mathcal{A}(E) \subseteq \mathcal{A}(E') \cup \mathcal{A}(E'')$.

Proof. If $E = 0 \Rightarrow g(E') = 0$.

$\Rightarrow E' = \text{Ker } g = 0 \Rightarrow E' = 0$.

That is $E = 0 \Rightarrow E' = 0$.

Also $E = 0 \Rightarrow f(E) = E'' = 0$. Therefore $E = 0 \Rightarrow E'' = 0$.

And therefore the result stated is trivially true.

Hence $\mathcal{A}(E') = \phi \subseteq \mathcal{A}(E) = \phi \subseteq \mathcal{A}(E') \cup \mathcal{A}(E'')$.

Assume $E \neq 0$.

g is one-one and E' can be considered as an H-sub hemiring group of E.

Therefore $\mathcal{A}(E') \subseteq \mathcal{A}(E) = \{\text{Ann}(P) |$ for some prime H-sub hemigroup P of E$\}$.

for some prime H-sub hemigroup E' of E.

Consider $A(P) \in \mathcal{A}(E)$ for some prime H-sub hemiring group P of E.

Two cases arises

Case-1 $E' \cap P \neq 0 \Rightarrow \text{Ann}(P) = \text{Ann}(E' \cap P)$

Now $E' \cap P \leq E'$

Therefore $\text{Ann}(E' \cap P) \in \mathcal{A}(E')$.

Now $\text{Ann}(P) \in \mathcal{A}(E')$.

Therefore $\mathcal{A}(E) \subseteq \mathcal{A}(E')$.

Case-2 $E' \cap P = 0$ and P(Prime) $\leq E \Rightarrow \mathcal{A}(E) \subseteq \mathcal{A}(E'')$.

Combining case-1 and 2 we have $\mathcal{A}(E) \subseteq \mathcal{A}(E') \cup \mathcal{A}(E'')$.

2.2.4. Lemma. Let N be an H-ideal of an H-hemiring group E such that A is closed H-sub hemiring group of E with $N \leq A \leq_c E$. Then $A/N \leq_c E/N$.

Proof. Let $A/N \leq E/N$ then $K/N \leq E/N$ such that $A/N \underset{\neq e}{\leq} K/N \leq E/N$. Then we have $A \lneqq K \leq E$. Also we know that there exists an H-epimorphisim

$f : K \to K/N,\ f(k) = k + n$

We note that $f^{-1}(A/N) = A$. Since $A/N <_e K/N$ it follows that $A <_e K$.

Since $K \leq E$, therefore $A \leq_e K \leq E \Rightarrow A = K$ (Since $A \leq_c E$).

$A/N \leq_e K/N \leq E/N \Rightarrow A/N = K/N$. Hence $A/N \leq_c E/N$.

$A/N \leq_e K/N \leq E/N \Rightarrow A/N = K/N$. Hence $A/N \leq_c E/N$.

2.2.5. Lemma. If E_1 and E_2 are two Goldie H-hemiring group then

$\mathcal{A}(E_1 \oplus E_2) = \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$

Proof. An arbitrary element of $A(E_1 \oplus E_2)$ is $A(Q)$ where Q is prime H-sub hemiring group of $E_1 \oplus E_2$.

Now $Q = Q_1 \oplus Q_2$ where $Q_1 \leq E_1$, $Q_2 \leq E_2$.

Since Q_1, Q_2 can be regarded as H-sub hemiring group of Q.

Therefore each of Q_1, Q_2 is prime and $A(Q_1) = A(Q_2) = A(Q)$.

it follows that $A(Q) \in \mathcal{A}(E_1), \mathcal{A}(E_2)$.

$\Rightarrow A(Q) \in \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

Therefore $\mathcal{A}(E_1 \oplus E_2) \subseteq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

To get the opposite relation

Let $A(Q) \in \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

Suppose $A(Q) \in \mathcal{A}(E_1)$. Then Q is a prime H-sub hemiring group of E_1 and hence a prime H-sub hemiring group of $E_1 \oplus E_2$.

So $A(Q) \in A(E_1 \oplus E_2)$.

Thus $\mathcal{A}(E_1) \cup \mathcal{A}(E_2) \subseteq \mathcal{A}(E_1 \oplus E_2)$.

This gives us the required result.

2.2.6. Lemma. If E is a Goldie H-hemi ring group and P and Q are H-sub hemiring groups of E, R is H-ideal of E such that $R \leq P, Q$ then

$\mathcal{A}(P \cap Q/R) = \mathcal{A}(P/R \cap Q/R) \subseteq \mathcal{A}(P/R) \cap \mathcal{A}(Q/R)$.

Proof. Let A(S/R) is an arbitrary element of $\mathcal{A}\left(\frac{P \cap Q}{R}\right)$ where S/R is a prime H-sub hemiring group of $\left(\frac{P \cap Q}{R}\right)$. Since $P \cap Q \leq P$, Q and $\left(\frac{P \cap Q}{R}\right)$ is contained in P/R and Q/R; hence in their intersection. Therefore the prime H-sub hemiring group S/R of $\frac{P \cap Q}{R}$ is also a prime H-sub hemiring group of the sub hemiring group $P/R \cap Q/R$.

Thus $A(S/R) \in \mathcal{A}(P/R \cap Q/R)$ which gives that $\mathcal{A}\left(\frac{P \cap Q}{R}\right) \subseteq \mathcal{A}(P/R \cap Q/R)$.

Conversely,

let $A(T/R) \in \mathcal{A}(P/R \cap Q/R)$ Where T/R is also a prime H-sub hemiring group of $P/R \cap Q/R$.

Then $T/R \leq P/R, Q/R$, it follows that $T \leq P$, Q and therefore $T \leq P \cap Q$.

We then have $T/R \leq \frac{P \cap Q}{R}$.

This means T/R is also a prime H-sub hemiring group of $\frac{P \cap Q}{R}$.

So $A(T/R) \in \mathcal{A}\left(\frac{P \cap Q}{R}\right)$

Therefore $\mathcal{A}(P/R \cap Q/R) \subseteq \mathcal{A}\left(\frac{P \cap Q}{R}\right)$

Therefore from above $\mathcal{A}(P/R \cap Q/R) = \mathcal{A}\left(\frac{P \cap Q}{R}\right)$.

Again for any $A(T/R) \in \mathcal{A}(P/R \cap Q/R)$,

T/R is a prime H-sub hemiring group of $(P/R \cap Q/R)$, we have $T/R \leq P/R, Q/R$; this means that T/R is a prime H-sub hemiring group of each of the H-hemiring group P/R and Q/R.

So $A(T/R)$ is contained in both $\mathcal{A}(P/R)$ and $\mathcal{A}(Q/R)$ and hence in their intersection. It follows; $\mathcal{A}(P/R \cap Q/R) \subseteq \mathcal{A}(P/R) \cap \mathcal{A}(Q/R)$.

2.2.7. Lemma. Let E_1 and E_2 be H-sub hemiring groups of an H-hemiring group E such

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that E_2 is essential in E_1. If $a \in E_1$, $a \neq 0$ then there is an essential H-sub hemiring group E' of E such that $E_2a \neq 0$, $E'a \subseteq E_2$.

Proof. $E' = \{h \in H \mid ha \in E_2\}$.

Let $h_1a, h_2a \in E'$.

Now $h_1a + h_2a = (h_1 + h_2)a = h_3a \in E'$ and $h'(ha) = (h'h)a = h_4a \in E'$ $(h' \in H)$.

Therefore E' is an H-sub hemiring group of E and $Ha \subseteq E_1$ (E_1 is an H-sub hemiring group of a H and $a \in E_1$).

Therefore $ha \in E_1$. Also $ha \neq 0$ (for $1 \in H$ implies $a \in Ha$)

Since $E_2 \leq_e E_1$, we get $Ha \cap E_2 \neq 0$.

Therefore consider $e_2(\in Ha) = ha\ (\neq 0) \in E_2$.

So $E'a \neq 0$.

We now show that $E' \leq_e E$.

Let I $(\neq 0)$ be an H-sub hemiring group of E. We claim that $I \cap E' \neq 0$.

Case-1: Suppose $Ia = 0$. Then $Ia \in E_2 \subseteq E_1$.

Therefore $Ia \subseteq E_1$.

So $I \subseteq E'$.

Hence $I \cap E' = I\ (\neq 0)$.

Case-2: Suppose $Ia \neq 0$.

Then $Ia \subseteq E_1$ since $E_2 \leq_e E_1$ $(I \leq H,\ a \in E_1 \subset H\ Ia \subseteq H)$,

$Ia \cap E_2 \neq 0$.

Hence for some $i\ (\neq 0) \in I$, $ia \in E_2\ (\subseteq E_1)$.

Thus $i \in E'$ (since $I \subseteq E'$).

Therefore $I \cap E' \neq 0$ which implies that E' is an essential H-sub hemiring group of E.

2.2.8. Lemma. Let P and Q be annihilators of subsets of E in H such that $P \subseteq Q$ and P is H-essential in Q. If $Z_1(E) = 0$ then $P = Q$.

Proof. Let $E_1, E_2 \subseteq E$.

$P = \mathrm{Ann}_H(E_1)$ $Q = \mathrm{Ann}_H(E_2)$ We have to show $Q \subseteq P$.

Let $q \in Q$ such that $q \neq 0$.

Since $P \subseteq Q$ and $P \leq_c Q$ there exists an essential H-sub hemiring group E′ of E such that $E'q \neq 0$, $E'q \subseteq P$ (lemma 2.2.7).

Therefore $A(E'q) \supseteq A(P)$

$\Rightarrow A(P) \subseteq A(E'q)$

$\Rightarrow E'qA(P) \subseteq E'\, q\, A\, (E'\, q) = 0.$

Therefore $E'\, q\, A(P) = 0.$

So $q\, A\, (P) = 0.$

$\Rightarrow$ $q \in Ann(A(P)) = P.$

Therefore $q \in P.$

Therefore $Q \subseteq P$ Hence $Q = P.$

EXAMPLE OF PRIME CLOSED, ESSENTIAL ETC.

Let $G = Z_{30}$ is Goldie Z-hemiring Group then its Z-sub hemiring Groups are

$A_1 = \{0, 2, 4, 6, 8, ... 28\}$, $A_2 = \{0, 3, 6, 9, 12, ... 27\}$, $A_3 = \{0, 5, 10, 15, ... 25\}$,

$A_4 = \{0, 6, 12, 18, 24\}$, $A_5 = \{0, 10, 20\}$, $A_6 = \{0, 15\}$

Now $A_4 < A_2 \lneqq_c G$. Similarly $A_5 < A_3 \lneqq_c G$ and $A_6 < A_3 \lneqq_c G$.

Now, $\frac{A_2}{A_4} = \{a_2 + A_4 \mid a_2 \in A_2\}$

$= \{0 + A_4, 3 + A_4, 6 + A_4, 9 + A_4 ... 27 + A_4\}.$

Now $0 + A_4 = \{0, 6, 12, 18, 24\} = A_4$; $1 + A_4 = \{1, 7, 13, 19, 25\}$; $2 + A_4 = \{2, 8, 14, 20, 26\}$; $3 + A_4 = \{3, 9, 15, 21, 27\}$; $4 + A_4 = \{4, 10, 16, 22, 28\}$; $5 + A_4 = \{5, 11, 17, 23, 29\}$; $6 + A_4 = \{6, 12, 18, 24, 0\} = A_4$; $9 + A_4 = \{9, 15, ... 27\} = 3 + A_4$

Again $G/A_4 = \{g + A_4 | g \in G\} = \{A_4, 1 + A_4, 2 + A_4, 3 + A_4, 4 + A_4, 5 + A_4\}$ and

$\frac{A_2}{A_4} = \{A_4, 3 + A_4\}$ contains two elements. Therefore order is 2.

Obviously $\frac{A_2}{A_4} < \frac{G}{A_4}$

Therefore $\frac{A_2}{A_4}$ has order 2 which is prime. So $\frac{A_2}{A_4}$ is a prime Z-sub hemiring group.

3. SECTION.

3.1. Main results.

3.1.1. Theorem. Let E be a Goldie H-hemiring group over hemiring H. If the set $A(E)$ is a union of two disjoint sets X and Y, then there exists a closed H-ideal E′ such that $\mathcal{A}(E') = X$ and $\mathcal{A}(E/E') = Y$.

Proof. Given $\mathcal{A}(E) = X \cup Y$ such that $X \cap Y = \phi$

Let $\mathcal{K} = \{N' \leq_c E \mid \mathcal{A}(N') \subseteq X\}$.

As 0 is a closed H-sub hemiring group of E and $\mathcal{A}(0) = \phi$,

we have $k \neq \phi$ ($\phi \subseteq X$). Since E is H-Goldie by lemma 2.2.1, $\mathcal{K}$ has a maximal element (say) E′.

Also, $X \cup \mathcal{A}(E/E') \supseteq X \cup Y$ (by lemma 2.2.3).

Since X and Y are disjoint sets, that is $X \cap Y = \phi$, therefore we have $Y \subseteq \mathcal{A}(E/E')$.

Suppose $\mathcal{A}(E/E') \supsetneqq Y$. Then there exists a prime H-sub hemiring group N'/E' of E/E' such that $A(N'/E') \in \mathcal{A}(E/E')$ and $A(N'/E') \notin Y$.

Now E is a Goldie H-hemiring group $A(N'/E') \in \mathcal{A}(E/E')$ and $A(N'/E') \notin Y$.

Moreover by hypothesis there is a prime closed H-extension T/E' such that

$E' \leq N' \leq T <_c E$ and $N'/E' \leq T/E' \leq_c E/E'$.

Thus T is a closed H-sub hemiring group of E.

Since T/E' is non-zero, $E' \subseteq T$ and $A(T/E') = A(N'/E')$. Since T/E is Prime, $\mathcal{A}(T/E')$ is a singleton set, say $\{P\}$. We write simply P. Thus $\mathcal{A}(T/E') = P$ and $P \notin Y$. Again by lemma 2.2.3.

$\mathcal{A}(T) \subseteq \mathcal{A}(E') \cup \mathcal{A}(T/E')$.

Since $\mathcal{A}(E') \subseteq X$ and $\mathcal{A}(T/E') = P$,

we get $\mathcal{A}(T) \subseteq X \cup P$.

Also $T \subseteq E$ and $\mathcal{A}(E) \subseteq X \cup Y$.

Since $\mathcal{A}(T) \subseteq \mathcal{A}(E)$, therefore $\mathcal{A}(T) \subseteq X \cup Y$.

So $P \notin Y$ gives $\mathcal{A}(T) \subseteq X$. Thus $T \in \mathcal{K}$.

and this contradicts the maximality of E. Therefore $\mathcal{A}(E/E') \subseteq Y$.

Thus $X \cup Y \subseteq \mathcal{A}(E') \cup Y$ and $X \cap Y = \phi$ gives $X \subseteq \mathcal{A}(E')$.

3.1.2. Theorem. Let E be a Goldie H-hemiring group as above. Then $\mathcal{A}(E)$ is finite.

Proof. We assume the opposite, that is, that $\mathcal{A}(E) = \{P, Q, R, \ldots\}$ is infinite.

If $\mathcal{A}(E) = P \cup Y$ and $P \notin Y$ (We write P for $\{P\}$)

Then by theorem 3.1.1 there exists a closed H-ideal E′ of E such that

$\mathcal{A}(E') = P$,

$\mathcal{A}(E/E') = Y$. Thus $\mathcal{A}(E) = P \cup Y$.

$\mathcal{A}(E) = A(E') \cup A(E/E')$.

Since $Q \in \mathcal{A}(E)$ we have $Q \in \mathcal{A}(E/E')$,

so for some prime H-sub hemiring group B'/E' of E/E', $A(B'/E') = Q$.

Thus $\mathcal{A}(B'/E') = Q$.

Thus by hypothesis there is a prime extension E'/E' such that

$B'/E' < E'/E' \leq_c E/E'$ and $E' \leq B' \leq E'' \leq_c E$. Hence $A(E'/E') = A(B'/E') = Q$.

Therefore $\mathcal{A}(E'/E') = \mathcal{A}(B'/E') = Q$.

And by lemma 2.2.3, $\mathcal{A}(E') \subseteq \mathcal{A}(E') \cup \mathcal{A}(E'/E')$. It follows $\mathcal{A}(E') \subseteq \{P, Q\}$.

Also by lemma 2.2.3, $\mathcal{A}(E) \subseteq \mathcal{A}(E') \cup \mathcal{A}(E/E')$. Therefore

$\mathcal{A}(E) \subseteq \{P, Q\} \cup \mathcal{A}(E/E')$, that is, $R \in \mathcal{A}(E/E')$.

in a like manner we get another closed H-sub hemiring group E‴ of E such that $E' < E'' < E'''$ and for $S \in \mathcal{A}(E)$, $S \in \mathcal{A}(E/E')$. Since $\mathcal{A}(E)$ is infinite we get a strictly ascending infinite sequence of closed H-sub hemiring group which contradicts the Goldie characters of E because of lemma 2.2.1.

Hence $\mathcal{A}(E)$ is finite.

3.1.3 Theorem. Let the H-hemiring group E be fully Goldie as above.

1) There exists an H-primary decomposition of 0 in E.

2) If $E_1, E_2, \ldots, E_t$ are H-ideals and $E_1 \cap E_2 \cap \ldots \cap E_t$ is an H-primary decomposition of 0 in E then

$$\mathcal{A}(E) = \mathcal{A}(E/E_1) \cup \mathcal{A}(E/E_2) \cup \ldots \cup \mathcal{A}(E/E_t).$$

Proof. by the above theorem, $\mathcal{A}(E)$ is finite.

Let $\mathcal{A}(E) = \{P_1, P_2, \ldots P_t\}$.

Since $\mathcal{A}(E)$ is expressible as union of two disjoint sets $\{P_1, P_2, \ldots \hat{P}_i \ldots P_t\}$ and $\{P_i\}$, by theorem 3.1.1, there exists a closed H-ideal $E_1, E_2, \ldots, E_t$ of E such that for each i, $\mathcal{A}(Ei) = \{P_1, P_2, \ldots \hat{P}_i \ldots P_t\}$ and $\mathcal{A}(E/E_i) = \{P_i\}$.

Also, for each i, E/E_i is H-primary and $\mathcal{A}(E/E_i) \neq \mathcal{A}(E/E_j)$ for $i \neq j$ and $\mathcal{A}(E_1 \cap E_2 \cap \ldots \cap E_t) \subseteq \mathcal{A}(E_1) \cap \ldots \cap \mathcal{A}(E_1)$. Since clearly $\mathcal{A}(E_1) \cap \mathcal{A}(E_2) \ldots \cap \mathcal{A}(E_t) = \phi$, we then have $\mathcal{A}(E_1 \cap \ldots \cap E_t) = \phi$ and therefore by Lemma 2.2.2, $E_1 \cap E_2 \cap \ldots \cap E_t = 0$.

If possible let $E_1 \cap E_2 \cap \ldots \cap \hat{E}_i \ldots \cap E_t = 0$, that is, $\bigcap_{j \neq i} E_j$ for some i, $1 \leq i \leq t$.

Then we get an H-homomorphism.

$\alpha : E \to \bigoplus_{j \neq i} E/E_j$, $e \to (e + E_1, \ldots e + \hat{E}_i, \ldots, e + E_t)$.

We note that ker $\alpha = \{e \mid e \in \bigcap_{i \neq j} E_j = 0\} = 0$.

Thus α is an embedding and hence $\mathcal{A}(E) \subseteq \mathcal{A}(\bigoplus_{j \neq i} E/E_j)$. Since E is fully Goldie, each E/E_j is Goldie. So it follows from Lemma 2.2.5 for each i,

$\mathcal{A}(E) \subseteq \bigcup_{i \neq j} \mathcal{A}(E/E_j)$, that is, $\mathcal{A}(E) \subseteq \{P_1, \ldots, \hat{P}_i, \ldots, P_t\}$ which is absurd. Hence $\bigcap_{i \neq j} E_j \neq 0$.

(ii) Next suppose that $\bigcap_{j=1}^{t}$ is an H-primary decomposition of 0 in E. Then the map

$\alpha : E \to \bigoplus_{j=1}^{t} E/E_j$, $e \to (e + E_1, e + E_2, \ldots e + E_t)$

is an embedding, which means that

$\mathcal{A}(E) \subseteq \mathcal{A}(\oplus E/E_j)$ and hence $\mathcal{A}(E) \subseteq \cup \mathcal{A}(E/E_j)$.

To see the opposite inclusion consider the H-homomorphism

$\beta : \bigcap_{i \neq j} E_j \to E/E_i$, $e \to e + E_i$.

Now ker $\beta = \{e \mid e \in \cap E_j\} = 0$. Thus $\mathcal{A}(\bigcap_{i \neq j} E_j) \subseteq \mathcal{A}(E/E_i)$ and by lemma 2.2.2,

$\mathcal{A}(\bigcap_{i\neq j} E_j) \neq \phi$. Since $\mathcal{A}(E/E_i)$ is singleton, we get $\mathcal{A}(\bigcap_{i\neq j} E_j) = \mathcal{A}(E/E_i)$ for each i, Hence

$$\bigcup_{j=1}^{l} \mathcal{A}(E/E_j) = \bigcup_{i=1}^{l} (\mathcal{A}(\bigcap_{i\neq j} E_j))$$

and Since $\mathcal{A}(\bigcap_{i\neq j} E_j) \subseteq \mathcal{A}(\mathrm{E})$ for each i, we finally have $\bigcup_{j=1}^{l} \mathcal{A}(E/E_j) \subseteq \mathcal{A}(\mathrm{E})$.

Thus $\bigcup_{j=1}^{l} \mathcal{A}(E/E_j) = \mathcal{A}(\mathrm{E})$.

We now give one result on a Goldie H-hemiring group when the operating set H is a left hemi ring with no infinite direct sum of sub H-hemiring group and $Z_1(E) = 0$. Theorem 7 of Oswald [6] follows as a corollary to the following result in the case of a regular left Goldie near-ring [2].

3.1.4. Theorem. Let E be a Quasi-subtractive Goldie H-hemiring group with $Z_1(E) = 0$ as above and such that an essential H-ideal of E is essential as an H-sub hemiring group also. Then the annihilators of subsets of E in H satisfies the d.c.c.

Proof. Let B = A(Y), C = A(X), X, Y $\subseteq$ E. Then if X $\subseteq$ Y we have A(Y) $\subseteq$ A(X). Then B $\subseteq$ C. Then by lemma 2.2.8, there exists an H-sub hemiring group D of E such that D $\subseteq$ C, B $\cap$ D = 0. Thus if in the descending chain $A(S_1) \supseteq A(S_2) \supseteq A(S_3) \supseteq \ldots$ we have $A(S_k) \supsetneq A(S_{k+1})$, then there exists H-sub hemiring group P_k such that $P_k \subseteq A(S_k)$ and $A(S_{k+1}) \cap P_k = 0$. Again we choose an H-sub hemiring group X_k such that $A(S_{k+1}) \cap X_k = 0$ and X_k is maximal for this.

Being the annihilators of S_{k+1} in E, $A(S_{k+1})$ is an H-ideal of E. So $A(S_{k+1}) + X_k$ is an H-sub hemiring group of E. So it is essential as an H-sub hemiring group.

Therefore $P_k \cap (A(S_{K+1}) + X_k) \neq 0$ (we write A_k for $A(S_k)$).

Now let $(0 \neq)\ b_k\ (\in P_k) = a_{k+1} + x_k,\ a_{k+1} \in A_{k+1},\ x_k \in X_k$. This implies

$0 = a_{k+1} + x_k + b_k'$ iff $x_k' = a_{k+1} + b_k' \in P_k + A_{k+1} \subseteq P_k + A_k$ (since $A_{k+1} \subseteq A_k$)

$\subseteq A_k \cap X_k$ (since $P_k \subseteq A_k$)

Now, $x_k' = 0 \Rightarrow a_{k+1} + b_k' = 0$

$\Rightarrow x_k = 0$

$$\begin{aligned} \Rightarrow b_k = a_{k+1} &\in P_k + A_{k+1} \\ &\subseteq P_k + A_k \text{ (since } A_{k+1} \subseteq A_k) \\ &\subseteq A_k \cap X_k \text{ (since } P_k \subseteq A_k) \end{aligned}$$

Now if x_k were 0, we would have $a_{k+1} = b_k \in P_k \cap A_{k+1} = 0$. So $x_k \neq 0$.

Therefore we get a non-zero H-ideal C_k and $C_k \cap A_{k+1} = 0$.

Therefore $x_k' \neq 0$.

An infinite descending chain of annihilators of subsets of E in H gives an infinite direct sum of H-ideals of E.

Since E has no infinite direct sum of H-ideals the descending chain $A_1 \supseteq A_2 \supseteq \ldots$ is a finite one.

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Jour. Pure Math., Vol. 23, 2006, pp. 37-52

MINIMAL STRUCTURES, CONTINUOUS MULTIFUNCTIONS AND BITOPOLOGICAL SPACES

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ABSTRACT : By using *m*-continuity between an *m*-space and a topological space, we establish the unified theory of several weak forms of continuity for multifunctions between bitopological spaces.

Key words and pharases : *m*-structure, *m*-open, (*i*, *j*)-*m*-continuous, *m*-continuous, bitopological space, multifunction.

AMS Subject Classification : 54C08; 54C60; 54E55.

1. INTRODUCTION

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researches of generalizations of continuity in topological spaces and bitopological spaces. By using these sets, many authors introduced and studied various types of modifications of continuity in topological spaces and bitopological spaces. Maheshwari and Prasad [13] and Bose [2] introduced the concepts of semi-open sets and semi-continuity in bitopo-logical spaces. Jelic [5], Kar and Bhattacharyya [7] and Khedr et al. [8] introduced the concepts of preopen sets and precontinuity in bitopological spaces. The notions of α-open sets (or feebly open sets) and α-continuity (or feeble continuity) in bitopological spaces were studied in [6], [18] and [11]. The notions of semi-preopen sets and semi-precontinuity in bitopological spaces were studied in [8].

Smithson [31] initiated the study of multifunctions in bitopological spaces. Several authors studied different forms of continuity for multifunctions in bitopological spaces in [3], [10], [12], [16], [17], [24], [25] and [26]. Recently, the present authors [27], [28] introduced and investigated the notions of minimal structures, *m*-spaces, *m*-continuity and *M*-continuity for functions and extended the concepts of *m*-continuity and *M*-continuity for multifunctions. In the present paper, we reduce the study of different forms of continuity for multifunctions between bitopological spaces to *m*-continuity between an *m*-space and a topological space. We present general theorems for characterizations and some properties of these forms of continuity between bitopological spaces. New forms of continuous multifunctions are initiated.

2. PRELIMINARIES

Throughout the present paper (X, τ) and (Y, σ) always denote topological spaces and (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote bitopological spaces. Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (or briefly *m-structure*) [27], [28] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Definition 2.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [14] as follows:

(1) $m\mathrm{Cl}(A) = \cap\{F : A \subset F,\ X - F \in m_X\}$,

(2) $m\mathrm{Int}(A) = \cup\{U : U \subset A,\ U \in m_X\}$.

Lemma 2.1. (Maki et al. [14]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $m\mathrm{Cl}(X - A) = X - m\mathrm{Int}(A)$ *and* $m\mathrm{Int}(X - A) = X - m\mathrm{Cl}(A)$,

(2) If $(X - A) \in m_X$, *then* $m\mathrm{Cl}(A) = A$ *and if* $A \in m_X$, *then* $m\mathrm{Int}(A) = A$,

(3) $m\mathrm{Cl}(\emptyset) = \emptyset$, $m\mathrm{Cl}(X) = X$, $m\mathrm{Int}(\emptyset) = \emptyset$ *and* $m\mathrm{Int}(X) = X$,

(4) If $A \subset B$, *then* $m\mathrm{Cl}(A) \subset m\mathrm{Cl}(B)$ *and* $m\mathrm{Int}(A) \subset m\mathrm{Int}(B)$,

(5) $A \subset m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(A) \subset A$,

(6) $m\mathrm{Cl}(m\mathrm{Cl}(A)) = m\mathrm{Cl}(A)$ *and* $m\mathrm{Int}(m\mathrm{Int}(A)) = m\mathrm{Int}(A)$.

Lemma 2.2. (Popa and Noiri [27]. *Let (X, m_X) be an m-space and A a subset of X. Then $x \in m\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 2.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [14] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 2.3. (Popa and Noiri [30]). *Let (X, m_X) be an m-space and m_X satisfy property $\mathcal{B}$. Then for a subset A of X, the following properties hold:*

(1) $A \in m_X$ *is and only if* $m\mathrm{Int}(A) = A$,

(2) *A is m-closed if and only if* $m\mathrm{Cl}(A) = A$,

(3) $m\mathrm{Int}(A) \in m_x$ *and* $m\mathrm{Cl}(A)$ *is m-closed.*

Definition 2.4. Let (X, m_x) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m_x) \to (Y, \sigma)$ is said to be

(1) *upper m-continuous* [29] if for each $x \in X$ and each $V \in \sigma$ *containing* $F(x)$, there exists $U \in m_x$ containing x such that $F(U) \subset V$,

(2) *lower m-continuous* [29] if for each $x \in X$ and each $V \in o$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_x$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Theorem 2.1. *For a multifunction* $F : (X, m_x) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) *F is upper m-continuous;*

(2) $F^+(V) = m\mathrm{Int}(F^+(V))$ *for every* $V \in \sigma$;

(3) $F^-(K) = m\mathrm{Cl}(F^-(K))$ *for every closed set K of Y;*

(4) $m\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Cl}(B)$ *for every subset B of Y;*

(5) $F^+(\mathrm{Int}(B)) \subset m\mathrm{Int}(F^+(B))$ *for every subset B of Y.*

Proof. The proof follows from Theorem 3.1 of [19] and Lemma 4.1 of [29].

Corollary 2.1. *Let* (X, m_x) *be an m-space and* m_x *satisfy property* B. *For a multifunction* $F : (X, m_x) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) *F is upper m-continuous;*

(2) $F^+(V)$ *is* m_x*-open for every open set V of Y;*

(3) $F^-(K)$ *is* m_x*-closed for every closed set K of Y.*

Proof. The proof follows from Theorem 2.1 and Lemma 2.3.

Theorem 2.2. *For a multifunction* $F : (X, m_x) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) *F is lower m-continuous;*

(2) $F^-(V) = m\mathrm{Int}(F^-(V))$ *for every* $V \in \sigma$;

(3) $F^+(K) = m\mathrm{Cl}(F^+(K))$ *for every closed set K of Y;*

(4) $m\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{Cl}(B)$ *for every subset B of Y;*

(5) $F^-(\mathrm{Int}(B)) \subset m\mathrm{Int}(F^-(B))$ *for every subset B of Y;*

(6) $F(m\mathrm{Cl}(A)) \subset \mathrm{Cl}(F(A))$ *for every subset* A *of* X.

Proof. The proof follows from Theorem 3.2 of [19] and Lemma 4.1 of [29].

Corollary 2.2. *Let* (X, m_X) *be an m-space and* m_X *satisfy property* $\mathcal{B}$. *For a multifunction* $F : (X, m_X) \rightarrow (Y, \sigma)$, *the following properties are equivalent.*

(1) F *is lower m-continuous;*

(2) $F^-(V)$ *is* m_X*-open for every open set* V *of* Y;

(3) $F^+(K)$ *is* m_X*-closed for every closed set* K *of* Y.

Proof. The proof follows from Theorem 2.2 and Lemma 2.3.

3. MINIMAL STRUCTURES AND BITOPOLOGICAL SPACES

First, we recall some definitions of weak forms of open sets in a bitopological space. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. The closure of A and the interior of A with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1, 2$.

Definition 3.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-semi-open* [13] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-preopen* [5] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-α-open* [6] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,

(4) *(i, j)-semi-preopen* [8] if there exists an *(i, j)*-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

The family of *(i, j)*-semi-open (resp. *(i, j)*-preopen, *(i, j)*-α-open, *(i, j)*-semi-preopen) sets of (X, τ_1, τ_2) is denoted by $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$).

Remark 3.1. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then $(i, j)\mathrm{SO}(X)$, $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$ and $(i, j)\mathrm{SPO}(X)$ are all *m*-structures on X. Hence, if $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$), then we have

(1) $m_{ij}\mathrm{Cl}(A) = (i, j)\text{-}\mathrm{sCl}(A)$ [13] (resp. $(i, j)\text{-}\mathrm{pCl}(A)$ [8], $(i, j)\text{-}\alpha\mathrm{Cl}(A)$ [18], $(i, j)\text{-}\mathrm{spCl}(A)$ [8]),

(2) $m_{ij}\mathrm{Int}(A) = (i, j)\text{-}\mathrm{sInt}(A)$ (resp. $(i, j)\text{-}\mathrm{pInt}(A)$, $(i, j)\text{-}\alpha\mathrm{Int}(A)$, $(i, j)\text{-}\mathrm{spInt}(A)$).

Remark 3.2. Let (X, τ_1, τ_2) be a bitopological space. Then

(1) Let $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\alpha(X)$). Then by Lemma 2.1 we obtain the results established in Theorem 13 of [13] (resp. Theorem 3.6 of [18]).

(2) Let $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 2.2 we obtain the results cstablished in Theorem 1.15 of [12] (resp. Theorem 3.5 of [8], Theorem 3.5 of [18], Theorem 3.5 of [8]).

Remark 3.3. Let (X, τ_1, τ_2) be a bitopological space. Then

(1) $(i, j)\mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$ is an m-structure on X satisfying property $(\mathcal{B})$ by Theorem 2 of [13] (resp. Theorem 4.2 of [7] or Theorem 3.2 of [8], Theorem 3.2 of [18], Theorem 3.2 of [8]).

(2) Let $m_{ij} = (i, j)\ \mathrm{SO}(X)$ (resp. $(i, j)\mathrm{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\mathrm{SPO}(X)$). Then, by Lemma 2.3 we obtain the results established in Theorem 1.13 of [12] (resp. Theorem 3.5 of [8], Theorem 3.6 of [18], Theorem 3.6 of [8]).

4. *m*-CONTINUITY AND BITOPOLOGICAL SPACES

Definition 4.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) *(i, j)-upper semi-continuous* or *(i, j)-upper quasi-continuous* [26] if for each $x \in X$ and each σ_i-open set V of Y containing $F(x)$, there exists an (i, j)-semi-open set U containing x such that $F(U) \subset V$;

(2) *(i, j)-lower semi-continuous* or *(i, j)-lower quasi-continuous* [26] if for each $x \in X$ and each σ_i-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an (i, j)-semi-open set U containing x such that $F(u) \cap V \neq \emptyset$, there exists an (i, j)-semi-open set U containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$;

(3) *pairwise upper / lower semi-continuous* or *pairwise upper / lower quasi-continuous* if it is (i, j)-upper/lower semi-continuous and (j, i)-upper/lower semi-continuous for $i, j = 1, 2$ and $i \neq j$.

Remark 4.1. If $m_{ij} = (i, j)\mathrm{SO}(X)$, then a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-upper/lower semi-continuous if and only if a multifunction $F : (X, m_{ij}) \to (Y, \sigma_i)$ is upper/lower m-continuous.

Definition 4.2. Let (X, τ_1, τ_2) be a bitopological space and m_{ij} and m_{ji} m-structures on X determined by τ_1 and τ_2. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-upper/lower m-continuous* if $F : (X, m_{ij}) \to (Y, \sigma_i)$ is upper/lower m-continuous. $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *pairwise upper/lower m-continuous* if $F : (X, m_{ij}) \to (Y, \sigma_i)$ and $F : (X, m_{ji}) \to (Y, \sigma_j)$ are upper/lower m-continuous for $i, j = 1, 2$ and $i \neq j$.

By Definition 4.2, Theorems 2.1 and 2.2, we obtain the following two theorems.

Theorem 4.1. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) F *is* (i, j)*-upper m-continuous;*

(2) $F^+(V) = m_{ij}\mathrm{Int}(F^+(V))$ *for every* $V \in \sigma$;

(3) $F^-(K) = m_{ij}\mathrm{Cl}(F^-(K))$ *for every closed set* K *of* Y;

(4) $m_{ij}\mathrm{Cl}(F^-(B)) \subset F^-(i\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^+(i\mathrm{Int}(B)) \subset m_{ij}\mathrm{Int}(F^+(B))$ *for every subset* B *of* Y,

where m_{ij} *is an m-structure determined by* τ_1 *and* τ_2.

Corollary 4.1. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) F *is* (i, j)*-upper m-continuous;*

(2) $F^+(V)$ *is* m_{ij}*-open for every* σ_i*-open set* V *of* Y;

(3) $F^-(K)$ *is* m_{ij}*-closed for every* σ_i*-closed set* K *of* Y,

where m_{ij} *is an m-structure satisfying property* $\mathcal{B}$ *which is determined by* τ_1 *and* τ_2.

Remark 4.2. If $m_{ij} = (i, j)\mathrm{SO}(X)$ (resp. $m_{ij} = (i, j)\mathrm{PO}(X)$), then by Theorem 4.1 and Corollary 4.1 we obtain the results established in Lemma 1 of [26] (resp. Theorem 4.2(a) of [16] and Theorem 2.9 of [3]).

Theorem 4.2. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) F *is* (i, j)*-lower m-continuous;*

(2) $F^-(V) = m_{ij}\mathrm{Int}(F^-(V))$ *for every* σ_i*-open set* V *of* Y;

(3) $F^+(K) = m_{ij}\mathrm{Cl}(F^+(K))$ *for every* σ_i*-closed set* K *of* Y;

(4) $m_{ij}\mathrm{Cl}(F^+(B)) \subset F^+(i\mathrm{Cl}(B))$ *for every subset* B *of* Y;

(5) $F^-(i\mathrm{Int}(B)) \subset m_{ij}\mathrm{Int}(F^-(B))$ *for every subset* B *of* Y;

(6) $F(m_{ij}\mathrm{Cl}(A)) \subset i\mathrm{Cl}(F(A))$ *for every subset* A *of* X,

where m_{ij} *is an m-structure determined by* τ_1 *and* τ_2.

Corollary 4.2. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, *the following properties are equivalent:*

(1) F *is* (i, j)*-lower m-continuous;*

(2) *$F^-(V)$ is m_{ij}-open for every σ_i-open set V of Y;*

(3) *$F^+(K)$ is m_{ij}-closed for every σ_i-closed set K of Y,*

where m_{ij} is an m-structure satisfying property $\mathcal{B}$ which is determined by τ_1 and τ_2.

Remark 4.3. If $m_{ij} = (i, j)SO(X)$ (resp. $m_{ij} = (i, j)PO(X)$), then by Theorem 4.2 and Corollary 4.2 we obtain the results established in Lemma 2 of [26] (resp. Theorem 4.2 of [16] and Theorem 3.7 of [3]).

5. SOME PROPERTIES

Definition 5.1. A subset B of a topological space (Y, σ) is said to be

(1) *α-regular* [9] if for each $b \in B$ and any open set U containing b, there exists an open set G of Y such that $b \in G \subset \mathrm{Cl}(G) \subset U$,

(2) *α-paracompact* [33] if every σ-open cover of B has a σ-open refinement which covers B and is locally finite for each point of Y.

For a multifunction $F : X \to Y$, by $\mathrm{Cl}(F) : X \to Y$ [1] we denote a multifunction defined as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, $\mathrm{sCl}(F)$, $\mathrm{pCl}(F)$, ${}_{\alpha}\mathrm{Cl}(F)$ and ${}_{\beta}\mathrm{Cl}(F)$ are defined, where $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, ${}_{\alpha}\mathrm{Cl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$) denotes the semi-closure (resp. preclosure, α-closure, β-closure) of a subset A of X.

Lemma 5.1. (Popa and Noiri [29]) *If $F : (X, m_{\tau}) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$, then $G^+(V) = F^+(V)$ for each open set V of Y, where G denotes $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$ or ${}_{\beta}\mathrm{Cl}(F)$.*

Lemma 5.2. (Popa and Noiri [29]) *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction. Then $G^-(V) = F^-(V)$ for each open set V of Y, where G denotes $\mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, ${}_{\alpha}\mathrm{Cl}(F)$ or ${}_{\beta}\mathrm{Cl}(F)$.*

Theorem 5.1. *Let $F : (X, m_X) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then the following properties are equivalent:*

(1) *F is upper m-continuous;*

(2) *$\mathrm{Cl}(F)$ is upper m-continuous;*

(3) *$\mathrm{sCl}(F)$ is upper m-continuous;*

(4) *$\mathrm{pCl}(F)$ is upper m-continuous;*

(5) *${}_{\alpha}\mathrm{Cl}(F)$ is upper m-continuous;*

(6) *${}_{\beta}\mathrm{Cl}(F)$ is upper m-continuous.*

Proof. We put $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, ${}_{\alpha}\mathrm{Cl}(F)$ or ${}_{\beta}\mathrm{Cl}(F)$ in the sequel.

Necessity. Suppose that F is upper m-continuous. Let $x \in X$ and V be any open set of Y such that $G(x) \subset V$; hence $x \in G^+(V)$. By Lemma 5.1 we have $x \in G^+(V) = F^+(V)$. Therefore, $F(x) \subset V$. Since F is upper m-continuous, there exists $U \in m_x$ containing x such that $F(U) \subset V$; hence $U \subset F^+(V) = G^+(V)$. We obtain $G(U) \subset V$. This shows that G is upper m-continuous.

Sufficiency. Suppose that G is upper m-continuous. Let $x \in X$ and V be any open sets of Y such that $F(x) \subset V$. Then, by Lemma 5.1 $x \in F^+(V) = G^+(V)$. Therefore, $G(x) \subset V$. Since G is upper m-continuous, there exists $U \in m_x$ containing x such that $G(U) \subset V$. Therefore, we obtain $x \in U \subset G^+(V) = F^+(V)$ and hence $F(U) \subset V$. Thus, F is upper m-continuous.

Corollary 5.1. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is σ_i-α-regular and σ_i-α-paracompact for each $x \in X$. Then the following properties are equivalent:*

(1) F is (i, j)-upper m-continuous;

(2) $\mathrm{Cl}(F)$ is (i, j)-upper m-continuous;

(3) $\mathrm{sCl}(F)$ is (i, j)-upper m-continuous;

(4) $\mathrm{pCl}(F)$ is (i, j)-upper m-continuous;

(5) $\alpha\mathrm{Cl}(F)$ is (i, j)-upper m-continuous;

(6) ${}_{\beta}\mathrm{Cl}(F)$ is (i, j)-upper m-continuous.

Proof. The proof follows from Definition 4.2 and Theorem 5.1.

Theorem 5.2. *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower m-continuous;

(2) $\mathrm{Cl}(F)$ is lower m-continuous;

(3) $\mathrm{sCl}(F)$ is lower m-continuous;

(4) $\mathrm{pCl}(F)$ is lower m-continuous;

(5) ${}_{\alpha}\mathrm{Cl}(F)$ is lower m-continuous;

(6) ${}_{\beta}\mathrm{Cl}(F)$ is lower m-continuous.

Proof. The proof is similar to that of Theorem 5.1.

Corollary 5.2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-lower m-continuous;*

(2) $\mathrm{Cl}(F)$ *is (i, j)-lower m-continuous;*

(3) $\mathrm{sCl}(F)$ *is (i, j)-lower m-continuous;*

(4) $\mathrm{pCl}(F)$ *is (i, j)-lower m-continuous;*

(5) ${}_{\alpha}\mathrm{Cl}(F)$ *is (i, j)-lower m-continuous;*

(6) ${}_{\beta}\mathrm{Cl}(F)$ *is (i, j)-lower m-continuous.*

Proof. The proof follows from Definition 4.2 and Theorem 5.2.

Definition 5.2. A bitopological space (X, τ_1, τ_2) is said to be *(i, j)-semi-compact* [4] if every cover of X by (i, j)-semi-open sets of X admits a finite subcover.

Definition 5.3. An m-space (X, m_X) is said to be *m-compact* [28] if every cover of X by m_X-open sets has a finite subcover. A subset K of (X, m_X) is said to be *m-compact* [28] if every cover of K by m_X-open sets has a finite subcover.

Definition 5.4. Let (X, τ_1, τ_2) be a bitopological space and m_{ij} an m-structure determined by τ_1 and τ_2. The space (X, τ_1, τ_2) is said to be *(i, j)-m-compact* if the m-space (X, m_{ij}) is m-compact. A subset K of X is said to be *(i, j)-m-compact* if K is m-compact in (X, m_{ij}).

Remark 5.1. Let (X, τ_1, τ_2) be a bitopological space and $m_{ij} = (i, j)\mathrm{SO}(X)$, then by Definition 5.3 we obtain the definition of (i, j)-semi-compact spaces.

Theorem 5.3. *Let $F : (X, m_X) \to (Y, \sigma)$ be upper m-continuous and $F(x)$ is compact for each $x \in X$. If K is an m-compact set of X, then $F(K)$ is compact.*

Proof. The proof follows from Theorem 4.1. of [19].

Corollary 5.3. *Let (X, τ_1, τ_2) be a bitopological space and m_{ij} an m-structure determined by τ_1 and τ_2. If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-upper m-continuous multifunction such that $F(x)$ is σ_i-compact for each $x \in X$ and K is (i, j)-m-compact, then $F(K)$ is σ_i-compact.*

Proof. Since F is (i, j)-upper m-continuous, $F : (X, m_{ij}) \to (Y, \sigma_i)$ is upper m-continuous and the corollary follows from Theorem 5.3.

Definition 5.5. A bitopological space (X, τ_1, τ_2) is said to be *pairwise-connected* [23] if X cannot expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open for $i \neq j$ and $i, j = 1, 2$.

Definition 5.6. Let (X, τ_1, τ_2) be a bitopological space with m-structures m_{ij} and m_{ji} on X determined by τ_1 and τ_2. The space (X, τ_1, τ_2) is said to be *pairwise m-connected* if X connot expressed as the union of two disjoint nonempty sets U and V such that $U \in m_{ij}$ and $V \in m_{ij}$ for $i \neq j$ and $i, j = 1. 2$.

Remark 5.2. Let (X, τ_1, τ_2) be a bitopological space. If $m_{ij} = (i, j)SO(X)$ and $m_{ji} = (j, i)SO(X)$, then we obtain the definition of pairwise semi-connected spaces due to Mukherjee [15].

Theorem 5.4. *Let (X, τ_1, τ_2) be a bitopological space with m-structures m_{ij} and m_{ji} determined by τ_1 and τ_2 which satisfy property $\mathcal{B}$. If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise upper or lower m-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, τ_1, τ_2) is pairwise m-connected, then (Y, σ_1, σ_2) is pairwise connected.*

Proof. Let (Y, σ_1, σ_2) be not pairwise connected. Then there exist a nonempty τ_i-open set U and a nonempty τ_j-open set V such that $Y = U \cup V$ and $U \cap V = \emptyset$. Since $F(x)$ is connected for each $x \in X$, by Theorem E of [23] $F(x) \subset U$ or $F(x) \subset V$. If $x \in X$, then $x \in F^+(U \cup V)$ and hence $x \in F^+(U) \cup F^+(V)$. Since F is surjective, there exist x and y in X such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following.

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,

(2) $F^+(U) \cap F^+(V) = \emptyset$ and

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ is (i, j)-m-open and $F^+(V)$ is (j, i)-m-open. (i) Let F be pairwise upper m-continuous. By Corollary 4.1, $F^+(U)$ is (i, j)-m-open and $F^+(V)$ is (j, i)-m-open. (ii) Let F be pairwise lower m-continuous. Since U is τ_j-closed, by Corollary 4.2, $F^+(U)$ is (j, i)-m-closed; hence $F^+(V)$ is (j, i)-m-open. Similarly, $F^+(U)$ is (i, j)-m-open. Consequently, X is not pairwise m-connected. This completes the proof.

6. THE SET OF POINTS OF UPPER/LOWER *m*-DISCONTINUITY

For a multifunction $F : (X, m_X) \to (Y, \sigma)$, we define $D_m^+(F)$ and $D_m^-(F)$ as follows:

$D_m^+(F) = \{x \in X : F$ is not upper m-continuous at $x\}$,

$D_m^-(F) = \{x \in X : F$ is not lower m-continuous at $x\}$.

Lemma 6.1. (Noiri and Popa [20]) *For a multifunction $F : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D_m^+(F) &= \cup_{G\in\sigma}\{F^+(G) - [m\mathrm{Int}(F^+(G))]\} \\ &= \cup_{B\in P(Y)}\{F^+(\mathrm{Int}(B)) - [m\mathrm{Int}(F^+(B))]\} \\ &= \cup_{B\in P(Y)}\{m\mathrm{Cl}(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \cup_{H\in F}\{m\mathrm{Cl}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where F is the family of closed sets of (Y, σ).

Lemma 6.2. (Noiri and Popa [20]) *For a multifunction* $F : (X, m_X) \to (Y, \sigma)$, *the following properties hold:*

$$\begin{aligned} D_m^-(F) &= \cup_{G \in \sigma}\{F^-(G) - [m\mathrm{Int}(F^-(G))]\} \\ &= \cup_{B \in P(Y)}\{F^-(\mathrm{Int}(B)) - [m\mathrm{Int}(F^-(B))]\} \\ &= \cup_{B \in P(Y)}\{m\mathrm{Cl}(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \cup_{A \in P(X)}\{m\mathrm{Cl}(A) - F^+(\mathrm{Cl}(F(A)))\} \\ &= \cup_{H \in \mathcal{F}}\{m\mathrm{Cl}(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}$ *is the family of closed sets of* (Y, σ).

By Definition 4.2, Lemmas 6.1 and 6.2, we obtain the following two theorems.

Theorem 6.1. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \tau_1, \tau_2)$ *a multifunction and* m_{ij} *be an m-structure determined by* τ_1 *and* τ_2. *Then, for a multifunction* $F : (X, m_{ij}) \to (Y, \sigma_i)$, *the following properties hold:*

$$\begin{aligned} D_{m_{ij}}^+(F) &= \cup_{G \in \sigma_i}\{F^+(G) - [m_{ij}\mathrm{Int}(F^+(G))]\} \\ &= \cup_{B \in P(Y)}\{F^+(i\mathrm{Int}(B)) - [m_{ij}\mathrm{Int}(F^+(B))]\} \\ &= \cup_{B \in P(Y)}\{m_{ij}\mathrm{Cl}(F^-(B)) - F^-(i\mathrm{Cl}(B))\} \\ &= \cup_{H \in F_i}\{m_{ij}\mathrm{Cl}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\mathcal{F}_i$ *is the family of closed sets of* (Y, σ_i)

Theorem 6.2. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *a multifunction and* m_{ij} *be an m-structure determined by* τ_1 *and* τ_2. *Then, for a multifunction* $F : (X, m_{ij}) \to (Y, \sigma_i)$, *the following properties hold:*

$$\begin{aligned} D_{m_{ij}}^-(F) &= \cup_{G \in \sigma_i}\{F^-(G) - [m_{ij}\mathrm{Int}(F^-(G))]\} \\ &= \cup_{B \in P(Y)}\{F^-(i\mathrm{Int}(B)) - [m_{ij}\mathrm{Int}(F^-(B))]\} \\ &= \cup_{B \in P(Y)}\{m_{ij}\mathrm{Cl}(F^+(B)) - F^+(i\mathrm{Cl}(B))\} \\ &= \cup_{A \in P(X)}\{m_{ij}\mathrm{Cl}(A) - F^+(i\mathrm{Cl}(F(A)))\} \\ &= \cup_{H \in F_i}\{m_{ij}\mathrm{Cl}(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}_i$ *is the family of closed sets of* (Y, σ_i).

Definition 6.1. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *m-continuous* [28] if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.

For a function $f : (X, m_X) \to (Y, \sigma)$, by $D_{m_X}(f)$ we denote the set of all points where f is not m-continuous. Then, we obtain the following corollary.

Corollary 6.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and m_{ij} an m-structure determined by τ_1 and τ_2. Then, for the function $f : (X, m_{ij}) \to (Y, \sigma_i)$, the following hold:*

$$\begin{aligned} D_{m_{ij}}(f) &= \cup_{G \in \sigma_i} \{f^{-1}(G) - [m_{ij}\mathrm{Int}(f^{-1}(G))]\} \\ &= \cup_{B \in P(Y)} \{f^{-1}(i\mathrm{Int}(B)) - [m_{ij}\mathrm{Int}(f^{-1}(B))]\} \\ &= \cup_{B \in P(Y)} \{m_{ij}\mathrm{Cl}(f^{-1}(B)) - f^{-1}(i\mathrm{Cl}(B))\} \\ &= \cup_{A \in P(X)} \{m_{ij}\mathrm{Cl}(A) - f^{-1}(i\mathrm{Cl}(f(A)))\} \\ &= \cup_{F \in F_i} \{[m_{ij}\mathrm{Cl}(f^{-1}(F))] - f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}_i$ is the family of closed sets of (Y, σ_i).

7. NEW FORMS OF *(i, j)-m*-CONTINUOUS MULTIFUNCTIONS

There are many modifications of open sets in topological spaces. First, we recall δ-closed sets and θ-closed sets due to Velicko [32]. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is a δ-cluster (resp. θ-cluster) point of A if $\mathrm{Int}\,(\mathrm{Cl}(V)) \cap A \neq \emptyset$ (resp. $\mathrm{Cl}(V) \cap A \neq \emptyset$) for every open set V containing x. The set of all δ-cluster (resp. θ-cluster) points of A is called the *δ-closure* (resp. *θ-closure*) of A and is denoted by $\mathrm{Cl}_\delta(A)$ (resp. $\mathrm{Cl}_\theta(A)$). If $A = \mathrm{Cl}_\delta(A)$ (resp. $A = \mathrm{Cl}_\theta(A)$), then A is said to be *δ-closed* (resp. *θ-closed*) [32]. The complement of a δ-closed (resp. θ-closed) set is said to be *δ-open* (resp. *θ-open*). The union of all δ-open (resp. θ-open) sets contained in A is called the *δ-interior* (resp. *θ-interior*) of A and is denoted by $\mathrm{Int}_\delta(A)$ (resp. $\mathrm{Int}_\theta(A)$).

Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. The δ-closure (resp. θ-closure) of A and the δ-interior (resp. θ-interior) of A with respect to τ_i are denoted by $i\mathrm{Cl}_\delta(A)$ (resp. $i\mathrm{Cl}_\theta(A)$) and $i\mathrm{Int}_\delta(A)$ (resp. $i\mathrm{Int}_\theta(A)$).

Definition 7.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-δ-semi-open* [21] if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-δ-preopen* [22] if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-δ-semipreopen* (simply *(i, j)-δ-sp-open*) if there exists an (i, j)-δ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Definition 7.2. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-θ-semi-open* if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-θ-preopen* if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\theta(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) *(i, j)-θ-semipreopen* (simply *(i, j)-θ-sp-open)* if there exists an (i, j)-θ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

Let (X, τ_1, τ_2) be a bitopological space. The family of (i, j)-δ-semi-open (resp. (i, j)-δ-preopen, (i, j)-δ-sp-open, (i, j)-θ-semi-open, (i, j)-θ-preopen, (i, j)-θ-sp-open) sets of (X, τ_1, τ_2) is denoted by $(i, j)\delta SO(X)$ (resp. $(i, j)\delta PO(X)$, $(i, j)\delta SPO(X)$, $(i, j)\theta SO(X)$, $(i, j)\theta PO(X)$, $(i, j)\theta SPO(X)$).

Remark 7.1. Let (X, τ_1, τ_2) be a bitopological space. The family $(i, j)\delta SO(X)$, $(i, j)\delta PO(X)$, $(i, j)\delta SPO(X)$, $(i, j)\theta SO(X)$, $(i, j)\theta PO(X)$ and $(i, j)\theta SPO(X)$ are all m-structures with property $(\mathcal{B})$.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we can define many new types of (i, j)-upper/lower m-continuous multifunctions. For example, in case $m_{ij} = (i, j)\delta SO(X)$ (resp. $(i, j)\delta PO(X)$, $(i, j)\delta SPO(X)$, $(i, j)\theta SO(X)$, $(i, j)\theta PO(X)$, $(i, j)\theta SPO(X)$) we can define new types of (i, j)-upper/lower m-continuous multifunctions as follows:

Definition 7.3. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-upper δ-semi-continuous* (resp. *(i, j)-upper δ-precontinuous, (i, j)-upper δ-sp-continuous)* if for each $x \in X$ and each σ_i-open set V of Y containing $F(x)$, there exists a (i, j)-δ-semi-open (resp. (i, j)-δ-preopen, (i, j)-δ-sp-open) set U of X containing x such that $F(U) \subset V$, equivalently if $F : (X, (i, j)\delta SO(X)) \to (Y, \sigma_i)$ (resp. $F : (X, (i, j)\delta PO(X)) \to (Y, \sigma_i)$, $F : (X, (i, j)\delta SPO(X)) \to (Y, \sigma_i)$) is upper m-continuous.

Definition 7.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-upper-θ-semi-continuous* (resp. *(i, j)-θ-precontinuous, (i, j)-θ-sp-continuous)* if for each $x \in X$ and each σ_i-open set V of Y containing $F(x)$, there exists a (i, j)-θ-semi-open (resp. (i, j)-θ-preopen, (i, j)-θ-sp-open) set U of X containing x such that $F(U) \subset V$, equivalently if $F : (X, (i, j)\theta SO(X)) \to (Y, \sigma_i)$ (resp. $F : (X, (i, j)\theta PO(X)) \to (Y, \sigma_i)$, $F : (X, (i, j)\theta SPO(X)) \to (Y, \sigma_i)$ is upper m-continuous.

Definition 7.5. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-lower δ-semi-continuous* (resp. *(i, j)-lower δ-precontinuous, (i, j)-lower δ-sp-continuous)* if for each $x \in X$ and each σ_i-open set V of Y $F(x)$ $V \neq \emptyset$, there exists a (i, j)-δ-semi-open (resp. (i, j)-δ-preopen, (i, j)-δ-sp-open) set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$, equivalently if $F : (X, (i, j)\delta SO(X)) \to (Y, \sigma_i)$ (resp. $F : (X, (i, j)\delta PO(X)) \to (Y, \sigma_i)$, F: $(X, (i, j)\delta SPO(X)) \to (Y, \sigma_i)$) is lower m-continuous.

Definition 7.6. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-lower θ-semi-continuous* (resp. *(i, j)-lower θ-precontinuous, (i, j)-lower θ-sp-continuous)* if for each

$x \in X$ and each σ_i-open set V of Y $F(x) \cap V \neq \emptyset$, there exists a (i, j)-θ-semi-open (resp. (i, j)-θ-preopen, (i, j)-θ-sp-open) set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$, equivalently if $F : (X, (i, j)\theta SO(X)) \to (Y, \sigma_i)$ (resp. $F : (X, (i, j)\theta PO(X)) \to (Y, \sigma_i)$, $F : (X, (i, j)\theta SPO(X)) \to (Y, \sigma_i)$) is lower m-continuous.

Conclusion. For multifunctions defined in Definitions 7.3 – 7.6 and also any (i, j)-upper/ lower m-continuous multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by using any m-structure m_{ij} on X determined by τ_1 and τ_2, we can apply the results established in Sections 4, 5 and 6.

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Jour. Pure Math., Vol. 23, 2006, pp. 53-64

ON FUZZY HOMOMORPHISM AND FUZZY ISOMORPHISM ON NEAR-RINGS

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ABSTRACT : Certain properties of fuzzy homomorphism and fuzzy isomorphism of near-rings are investigated. The decomposition theorem for fuzzy homomorphism and isomorphism is proved and using the concept of π-fuzzy quotient near-rings the order preserving correspondence theorem between fuzzy ideals of near-rings is obtained.

Key words : Near-rings, fuzzy homomorphism, fuzzy isomorphism, fuzzy ideals, fuzzy left N-subgroups.
2000 AMS Classification : $54A20$, $54D30$.

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [12]. Since than these ideas have been applied to other algebraic structures like groups, rings, modules, topologies and so on. The notions of fuzzy sub near-rings and ideals were introduced by S. Abou-Zaid in 1991 [1] Extensive works in the field of fuzzy near-rings have been carried out by many researchers [3, 4, 5, 6]. In our continuous afford [8, 9, 10, 11], we have studied various aspects of fuzzy substructures of near-rings and near-ring groups. Chakraborty et al [2] introduced the notion of fuzzy homomorphism and proved an analogue of the fundamental theorem of homomorphism of groups. In this paper, we generalize the concept of fuzzy homomorphism and fuzzy isomorphism to near-rings and investigate certain properties. The decomposition theorem for fuzzy homomorphism and isomorphism is proved and the concept of π-fuzzy quotient near-ring leads to the order preserving correspondence theorem between fuzzy ideals of near-rings.

2. DEFINITIONS AND NOTATIONS

A *right near-ring* N is a system with two binary operations "+" and "." such that (i) (N,+) forms a group, not necessarily abelian, (ii) (N,.) forms a semi group and (iii) $(x+y)z = xz + yz$, for all $x, y, z \in N$.

A near-ring N is said to be *zero symmetric* if $x.0 = 0$, for all $x \in N$.

Throughout the paper, unless otherwise specified, N will denote a zero symmetric right near-ring with unity. S(N) denotes the class of fuzzy subsets of N, F(N), the class of fuzzy left N-subgroups of N.

Let $\mu \in S(N)$. Then μ is called a *fuzzy left (right) N-subgroup* [1] of N if for x, y $\in$ N, (i) $\mu(x - y) \geq \text{Min}\{\mu(x), \mu(y)\}$ and (ii) $\mu(xy) \geq \mu(y)$ ($\mu(xy) \geq \mu(x)$).

μ is called fuzzy N-subgroup if μ is both left and right fuzzy N-subgroup of N.

A fuzzy subset μ of N is called a *fuzzy left ideal* [1] of N if (i) μ is fuzzy subgroup of N, (ii) $\mu(x+y) = \mu(y+x)$, (iii) $\mu[a(b+x) - ab] \geq \mu(x)$, for a, b, x, y in N.

A fuzzy subset μ of N is called a *fuzzy right ideal* [1] of N if (i) μ is fuzzy subgroup of N, (ii) $\mu(x+y) = \mu(y+x)$, (iii) $\mu(xy) \geq \mu(x)$, for all x, y in N.

If μ is both fuzzy right as well as fuzzy left ideal of N then μ is called a *fuzzy ideal* of N.

Let $A \subseteq N$ and $t \in [0, 1]$. We define $A_t \in S(N)$ such that

$$A_t(x) = t, \text{ if } x \in A$$
$$= 0, \text{ if } x \notin A.$$

If $A = \{x\}$ then x_t is called a *fuzzy point*. Fuzzy subset χ_0 is defined by $\chi_0(x) = 1$ for $x = 0$, $\chi_0(x) = 0$ for $x \neq 0$ is called *zero fuzzy subset* of N.

Let $\mu, \theta \in S(N)$. Then the sum $\mu + \theta$, the product $\mu_0\theta$ and the unary operation $-\mu$ are defined as follows: For each $x \in N$

$$(\mu + \theta)(x) = \sup_{x = a + b}\{\min\{\mu(a), \theta(b)\}$$

$$(\mu_0\theta)(x) = \sup_{x = ab}\{\min\{\mu(a), \theta(b)\}$$

$$(-\mu)(x) = \mu(-x).$$

Let μ be a fuzzy subset of a near ring N. Then the set $\{x \in N: \mu(x) > 0\}$ is called *support* of μ and is denoted by μ^*. If $1 \in \text{Im}\mu$ then μ is called a *unitary* fuzzy subset. In case μ^* is finite then μ is called a finite fuzzy subset. The set $\{x \in N | \mu(x) \geq t\}$ is called the *level set of* μ and is denoted by μ_t. For a fuzzy point x_t of N, x is called *support* of x_t; denoted by $\text{supp}(x_t)$ and t is called *height* of x_t, denoted by $\text{hgt}(x_t)$. The class of all fuzzy points of N is denoted by $\rho(N)$ and $\rho(\mu)$ denotes the set $\{x_t : x_t \in \mu\}$.

Let N and K be two near rings, μ and θ be two fuzzy subsets of N and K respectively. Let $F : \rho(N) \to \rho(K)$ be a mapping. Then fuzzy subsets $F(\mu) : K \to [0, 1]$ and $F^{-1}(\theta) : N \to [0,1]$ are defined as follows:

$$F(\mu) = \cup\{F(x_t) : x_t \in \rho(\mu)\}$$

$F^{-1}(\theta) = \{x_t : F(x_t) \in \rho(\theta)\}$

$F(\mu)$ and $F^{-1}(\theta)$ are called *image of* μ and *inverse image of* θ respectively under F.

Let μ and θ be two fuzzy left N-subgroups of N. Then a mapping

$F : \rho(\mu) \to \rho(\theta)$ is called *fuzzy homomorphism* of the fuzzy left N-subgroups μ and θ if for all $x_t, y_s \in \rho(\mu)$

(i) $F(x_t + y_s) = F(x_t) + F(y_s)$.

(ii) $F(x_t y_s) = F(x_t) F(y_s)$.

A fuzzy homomorphism F is called *fuzzy isomorphism* if F is bijective. We write $\mu \cong \theta$, if there exists a fuzzy isomorphism $F : \rho(\mu) \to \rho(\theta)$.

3. PRELIMINARIES

Lemma 2.1. [1] A fuzzy subset μ of N is a fuzzy left N-subgroup if and only if the level subset μ_t, $t \in \mathrm{Im}\mu$ is a left N-subgroup of N.

The following lemma is straightforward so we omit the proof.

Lemma 2.2. Let $x_t, y_s \in \rho(N)$. Then (i) $x_t + y_s = (x + y)_{t\wedge s}$ where $t \wedge s = \min\{t, s\}$ (ii) $x_t \circ y_s = (xy)_{t\wedge s}$, (iii) $-(x_t) = (-x)_t$

Lemma 2.3. Let $\mu \in S(N)$. Then $\mu \in F(N)$ if and only if the following conditions hold.

(i) $x_t + y_s \in \rho(\mu)$, for all $x_t, y_s \in \rho(\mu)$

(ii) $x_t . y_s \in \rho(\mu)$, for all $x_t, y_s \in \rho(\mu)$

(iii) $-x_t \in \rho(\mu)$, for all $x_t \in \rho(\mu)$.

Note 1. It is to be noted that $(\rho(\mu), +, o)$ forms a semi near-ring which is termed as *fuzzy semi-near-ring.*

Lemma 2.4. Let x_t and y_s be two fuzzy points of N. Then the following conditions hold:

(i) $nx_t = (nx)_t$ for all $n \in N$.

(ii) $-(x_t) = (-x)_t$

Lemma 2.5. Let N and K be two near rings. Let μ and θ be two fuzzy left N-subgroups of N and K respectively. If $F : \rho(\mu) \to \rho(\theta)$ is a fuzzy homomorphism then for every $x_t, y_s \in \rho(\mu)$,

(i) supp $[F(0_t)] = \mathbf{0}$, where 0 and $\mathbf{0}$ are zeros of N and K respectively, and $t \in (0, \mu(0)]$.

(ii) $F(-x_t) = -F(x_t)$

(iii) $F(x_t^{-1}) = [F(x_t)]^{-1}$

(iv) $\mathrm{hgt}[F(0_t)] = \mathrm{hgt}[F(x_t)]$

(v) $\mathrm{supp}\ [F(x_t)] = \mathrm{supp}[F(x_s)]$

(vi) $\mathrm{hgt}[F(x_t)] \leq \mathrm{hgt}[F(x_s)]$ if $t \leq s$.

Proof. Let $x_t \in \rho(\mu)$ such that $F(x_t) = y_r$ where t $(0, \mu(0)]$ and $r \in (0, \theta(\mathbf{0})]$. Now

(i) $F(x_t) = F(x_t + 0_t)$ implies $y_r = y_r + z_p$, where $F(0_t) = z_p$

Thus $\mathbf{0}_r = \mathbf{0}_r + z_p$ as $- y_r \in \rho(\mu)$. This gives $\mathbf{0}_r = z_{r \wedge p}$ and hence

$z = \mathbf{0}$ and this gives that $\mathrm{supp}[F(0_t)] = \mathbf{0}$.

(ii) Let $F(0_t) = \mathbf{0}_s$ where $s \in (0, \theta(0)]$. Then it is easy to prove that $r \leq s$. Thus $\mathrm{hgt}[F(x_t)] \leq \mathrm{hgt}[F(0_t)]$.

Now $F(x_t + (-x_t)) = F(0_t)$ implies $y_r + w_u = \mathbf{0}_s$, where $F(-x_t) = w_u$, $r, u \leq s$ ...(A)

Thus $\mathbf{0}_r + w_u = -y_r + \mathbf{0}_s$ implies $w_{r \wedge u} = -y_r$. Hence $w = -y$. Thus from (A), we have $y_r + (-y)_u = \mathbf{0}_s \Rightarrow \mathbf{0}_{r \wedge u} = \mathbf{0}_s$. This gives $r = s$ or $u = s$. If $r = s$ and $u \neq s$ then $u > s$ which contradicts (A). Similarly $r \neq s$ and $u = s$ gives us a contradiction.

Hence $r = u = s$. Thus $w_u = -y_r$. Therefore $F(-x_t) = -F(x_t)$.

(iii) the proof is straightforward.

(iv) Let $F(0_t) = \mathbf{0}_s$. Then $F(x_t + (-x_t)) = \mathbf{0}_s$ for any $x_t \in \rho(\mu)$. This implies $F(x_t) + F(-x_t) = \mathbf{0}_s$ which gives $y_r - y_r = \mathbf{0}_s$, by (ii) where $F(x_t) = y_r$. Thus $r = s$.

This means $\mathrm{hgt}[F(0_t)] = \mathrm{hgt}[F(x_t)]$.

(v) If $t = s$ then nothing to prove. Let $s > t$. Then we have, $F(x_s + 0_t) = F(x_t)$. This gives $z_p + \mathbf{0}_r = F(x_t)$ where $F(x_s) = z_p$. Thus $z_{p \wedge s} = F(x_t)$. This shows that supp $[F(x_t)] = z = \mathrm{supp}\ [F(x_s)]$. similarly for $t > s$, the result holds.

(vi) Let $t \leq s$. Then $F(x_t) = y_r$ implies $y_r + \mathbf{0}_v = y_r$, where $F(0_s) = \mathbf{0}_v$. Thus $y_{r \wedge v} = y_r$ and it gives $r \leq v$. Hence $\mathrm{hgt}[F(x_t)] \leq \mathrm{hgt}[F(0_s)] = v = \mathrm{hgt}[F(x_s)]$, by (iv)

4. MAIN RESULTS

Now we prove our main results:

Theorem 3.1. Let N and K be two near rings. Let μ and θ be two fuzzy left N-subgroups of N and K respectively. A mapping $F : \rho(\mu) \to \rho(\theta)$ is a fuzzy homomorphism if and only if there exists an ordinary homomorphism $f : \mu^* \to \theta^*$ and an increasing function $\pi : (0, \mu(0)] \to (0, \theta(0)]$ such that for every $x_t \in \rho(\mu)$, $F(x_t) = [f(x)]_{\pi(t)}$.

Proof. Let $F : \rho(\mu) \to \rho(\theta)$ be a fuzzy homomorphism. A mapping $f : \mu^* \to \theta^*$ is defined by $f(x) = \text{supp}\ [F(x_{\mu(x)})]$, for all $x \in \mu^*$. Also we define $\pi : (0, \mu(0)] \to (0, \theta(0)]$ such that $\pi(t) = \text{hgt}\ [F(0_t)]$, for all $t \in (0, \mu(0)]$. Then

$$\text{supp}[F(x_t)] = \text{supp}[F(x_{\mu(x)})] = f(x) \qquad [\text{by } 2.5 \text{ (v)}]$$

and $$\text{hgt}[F(x_t)] = \text{hgt}[F(0_t)] = \pi(t) \qquad [\text{by } 2.5 \text{ (iv)}]$$

Thus $F(x_t) = [f(x)]_{\pi(t)}$.

By lemma 2.5 (vi), it is clear that π is an increasing function. Next let $x, y \in \mu^*$ and $\mu(x) = t$, $\mu(y) = s$.

Now, $[f(x + y)]_{\pi(t)} = F((x + y)_t) = F(x_t + y_t) = F(x_t) + F(y_t) = [f(x)]_{\pi(t)} + [f(y)]_{\pi(t)}$

This implies $[f(x + y)]_{\pi(t)} = [f(x) + f(y)]_{\pi(t)}$

Thus $f(x + y) = f(x) + f(y)$.

Also $[f(xy)]_{\pi(t)} = F((xy)_t) = F(x_t y_t) = F(x_t)F(y_t) = [f(x)]_{\pi(t)}\ [f(y)]_{\pi(t)}$

Thus $[f(xy)]_{\pi(t)} = [f(x)f(y)]_{\pi(t)}$

And it gives $f(xy) = f(x)f(y)$. Hence f is a near ring homomorphism.

Conversely let $x_t, y_s \in \rho(\mu)$. Then $F(x_t + y_s) = F\{(x + y)_{t \wedge s}\}$. It implies $F(x_t + y_s) = [f(x + y)]_{\pi(t \wedge s)} = [f(x) + f(y)]_{\pi(t) \wedge \pi(s)} = [f(x)]_{\pi(t)} + [f(y)]_{\pi(s)} = F(x_t) + F(y_s)$

Similarly it can be proved that $F(x_t y_s) = F(x_t)F(y_s)$. Thus $F : \rho(\mu) \to \rho(\theta)$ is a fuzzy homomorphism.

Theorem 3.2. Let N and K be two near rings. Let μ and θ be two fuzzy left N-subgroups of N and K respectively. A mapping. $F : \rho(\mu) \to \rho(\theta)$ is a fuzzy isomorphism if and only if there exists an ordinary isomorphism $f : \mu^* \to \theta^*$ and an strictly increasing onto function $\pi : (0, \mu(0)] \to (0, \theta(0)]$ such that for every $x_t \in \rho(\mu)$,

$$F(x_t) = [f(x)]_{\pi(t)}.$$

Proof. Let $F : \rho(\mu) \to \rho(\theta)$ is a fuzzy isomorphism. We define $f : \mu^* \to \theta^*$ and $\pi : (0, \mu(0)] \to (0, \theta(0)]$ as in theorem 3.1. Then it is sufficient to prove that f is bijective and π is strictly increasing onto function.

Let $x, y \in \mu^*$ such that $f(x) = f(y)$. Then $\text{supp}[F(x_{\mu(x)})] = \text{supp}[F(y_{\mu(y)})]$. By lemma 2.5(v) we get $\text{supp}[F(x_u)] = \text{supp}[F(y_u)]$, where $u \in (0, \mu(0)]$. Also we have, $\text{hgt}[F(x_u)] = \text{hgt}[F(0_u)] = \text{hgt}[F(y_u)]$, by lemma 2.5.(iv). Therefore $F(x_u) = F(y_u)$ and hence $x = y$. Thus f is one-one.

Next let y be any element of θ^*. Then $\theta(y) > 0$. So let $\theta(y) = v$ and so $y_v \in \rho(\theta)$ Since F is onto, there exists $x_u \in \rho(\mu)$ such that $F(x_u) = y_v$. Thus $[f(x)]_{\pi(u)} = y_v$ and hence $f(x) = y$. It proves that f is onto.

Now let t, s $\in (0, \mu(0)]$ such that $\pi(t) = \pi(s)$. Then $\mathrm{hgt}[F(x_t)] = \mathrm{hgt}[F(x_s)]$

Also we have, $\mathrm{supp}[F(x_t)] = \mathrm{supp}[F(x_s)]$. Hence $F(x_t) = F(x_s)$. This gives $t = s$

Consequently π is one-one and by theorem 3.1., π is increasing. It follows that π is strictly increasing function.

Again we claim that π is onto. Let $u \in (0, \theta(0)]$ be any point. Let y $\in$ K such that $\theta(y) \geq$ u. Then $y_u \in \rho(\theta)$. F being onto, there exists $x_t \in \rho(\mu)$, $t \in (0, \mu(0)]$ such that $F(x_t) = y_u$. Thus $[f(x)]_{\pi(t)} = y_u$. It implies $\pi(t) = u$ and hence π is onto

Conversely let $x_t, y_s \in \rho(\mu)$ such that $F(x_t) = F(y_s)$. Then $[f(x)]_{\pi(t)} = [f(y)]_{\pi(s)}$. This gives $f(x) = f(y)$ and $\pi(t) = \pi(s)$. Thus $x = y$ and $t = s$ and hence $x_t = y_s$. This proves that F is one-one

Next let $y_u \in \rho(\theta)$. Then $\theta(y) \geq$ u, where $y \in \theta^*$ and $u \in (0, \theta(0)]$.

Therefore there exist x $\in \mu^*$ and $t \in (0, \mu(0)]$ such that $f(x) = y$ and $\pi(t) = u$. Thus we have $y_u = [f(x)]_{\pi(t)} = F(x_t)$, where $x_t \in \rho(\mu)$. Hence F is onto and by theorem 3.1., F is a fuzzy isomorphism. This completes the proof.

Note 2. Since a fuzzy homomorphism F : $\rho(\mu) \to \rho(\theta)$ of two fuzzy left N-subgroups μ and θ of N may be defined by a corresponding homomorphism $f : \mu^* \to \theta^*$ and an increasing function $\pi : (0, \mu(0)] \to (0, \theta(0)]$, we use the symbol f_π for F.

Corollary 3.3. A mapping $f_\pi : \rho(N) \to \rho(K)$ where N, K are near rings is a fuzzy homomorphism if and only if there exists an ordinary homomorphism $f : N \to K$ and an increasing function $\pi : (0, 1] \to (0, 1]$ such that $\pi(t) = t$ for every $t \in (0, 1]$ such that $f_\pi(x_t) = [f(x)]_t$

Theorem 3.4. Let $f : N \to K$ be a mapping where N, K are near rings and $\pi : (0, 1] \to (0, 1]$ be an increasing function. Let $f_\pi : \rho(N) \to \rho(K)$ be defined by $f_\pi(x_t) = [f(x)]_{\pi(t)}$. Then for any fuzzy subset μ of N,

$$
\begin{aligned}
f_\pi(\mu)(y) &= \vee\{\pi(\mu(x))\}, && \text{if } x \in f^{-1}(y) \\
&= 0 && \text{if } f^{-1}(y) \text{ is empty.}
\end{aligned}
$$

And also if $\pi : (0, 1] \to (0, 1]$ is strictly increasing surjection and θ any fuzzy subset of K then,

$$
\begin{aligned}
f_\pi^{-1}(\theta)(x) &= \pi^{-1}(\theta(f(x))), && \text{if } f(x) \in \theta^* \\
&= 0 && \text{if } f(x) \notin \theta^*
\end{aligned}
$$

Proof. Let y be any element of K. Then

$$\begin{aligned} f_\pi(\mu)(y) &= [\cup\{f_\pi(x_t) : x_t \in \rho(\mu)\}](y) = \vee\{[f(x)]_{\pi(t)}(y) : \mu(x) \geq t\} && \text{..... (A)} \\ &= \vee\{\pi(t) : \mu(x) \geq t\}, && \text{if } x \in f^{-1}(y) \\ &= \vee\{\pi(u) : \mu(x) = u\}, && \text{if } x \in f^{-1}(y) \\ &= \vee\{\pi(\mu(x))\}, && \text{if } x \in f^{-1}(y) \end{aligned}$$

Also from (A) it is obvious that if $f^{-1}(y)$ is empty then $f_\pi(\mu)(y) = 0$.

Next let n be any element of N. Then,

$$\begin{aligned} f_\pi^{-1}(\theta)(n) &= [\cup\{x_t : f_\pi(x_t) \in \rho(\theta)\}](n) = \vee\{x_t(n) : [f(x)]_{\pi(t)} \in \rho(\theta)\} \\ &= \vee\{x_t(n) : \theta(f(x)) \geq \pi(t)\} = \vee\{u : \theta(f(x)) = \pi(u),\ x = n\} && \text{..... (B)} \end{aligned}$$

If $f(n) \in \theta^{*}$ then $f_\pi^{-1}(\theta)(n) = \vee\{u : \pi^{-1}[\theta(f(n))] = u\} = \pi^{-1}[\theta(f(n))]$

if $f(x) \notin \theta^{*}$ then from (B), we have $f_\pi^{-1}(\theta)(n) = 0$.

Theorem 3.5. Let N and K be two near rings and mapping $f_\pi : \rho(N) \to \rho(K)$ is a fuzzy homomorphism. Let μ and and θ be two fuzzy left N-subgroups of N and K respectively.

Then

(i) $f_\pi(\mu)(\mathbf{0}) = \pi(\mu(0))$

(ii) $\mu \subseteq \eta \Rightarrow f_\pi(\mu) \subseteq f_\pi(\eta)$ where η is fuzzy invariant sub near ring of N.

(iii) $\theta \subseteq \nu \Rightarrow f_\pi^{-1}(\theta) \subseteq f_\pi^{-1}(\nu)$, ν is a fuzzy invariant sub near ring of K.

(iv) $f_\pi(\mu \circ \eta) = f_\pi(\mu) \circ f_\pi(\eta)$

Proof. (i) We have $\mu(0) \geq \mu(x)$ for all $x \in$ N. Now

$$f_\pi(\mu)(\mathbf{0}) = \vee\{\pi(\mu(x)) : f(x) = \mathbf{0}\} = \pi(\mu(0)) \text{ since } f(0) = \mathbf{0}.$$

(ii) Let y be any element of K. Then

$$f_\pi(\mu)(y) = \vee\{\pi(\mu(x)) : f(x) = y\} \leq \vee\{\pi(\eta(x)) : f(x) = y\} = f_\pi(\eta)(y).$$

Also if $f^{1}(y)$ is empty then $f_\pi(\mu)(y) = 0 = f_\pi(\eta)(y)$.

(iii) Let x be any element of N. Then if $f(x) \in \theta^{*}$,

$$f_\pi^{-1}(\theta)(x) = \pi^{-1}[\theta(f(x))] \leq \pi^{-1}[\nu(f(x))] = f_\pi^{-1}(\nu)(x)$$

Also if $f(x) \notin \theta^{*}$, then $f_\pi^{-1}(\theta)(x) = 0 = f_\pi^{-1}(\nu)(x)$. Hence we get the required result.

(iv) Let y be any element of K. Then

$$f_\pi(\mu o \eta)(y) = \vee\{\pi(\mu o \eta)(x)) : f(x) = y\} = \vee[\pi\{\vee_{x=ab}\{\mu(a) \wedge \eta(b)\}\} : f(ab) = y]$$

$$= \vee[\pi\{\vee_{y=ab}\{\mu(a) \wedge \eta(b)\}\} : f(a)f(b) = pq = y],\ f(a) = p,\ f(b) = q$$

$$= \vee[\{\vee_{f(a)=p}\pi(\mu(a))\} \wedge \{\vee_{f(b)=q}\pi(\eta(b))\} : y = pq] = \vee[f_\pi(\mu)(p) \wedge f_\pi(\eta)(q) : y = pq]$$

$$= [f_\pi(\mu) \circ f_\pi(\eta)](y)$$

Also if $f^{-1}(y)$ is empty then $f_\pi(\mu o \eta)(y) = 0\ [f_\pi(\mu) \circ f_\pi(\eta)](y)$. Thus the result follows.

Theorem 3.6. Let mapping $f_\pi : \rho(N) \to \rho(K)$ where N, K are near rings be a fuzzy homomorphism and μ be a fuzzy invariant sub near ring (ideal) of N. Then $f_\pi(\mu)$ is fuzzy invariant sub near ring (ideal) of K if f_π is onto.

Proof. Let $f_\pi : \rho(N) \to \rho(K)$ be a fuzzy epimorphism. Then there exists an ordinary epimorphism $f : N \to K$ and an increasing surjection $\pi : (0, 1] \to (0, 1]$ such that $f_\pi(x_t) = [f(x)]_{\pi(t)}$. Let $a, b \in K$. Then there exists $m, n \in N$ such that $f(m) = a, f(n) = b$ which gives $f(m + n) = a + b$ and $f(mn) = ab$. Now,

$$f_\pi(\mu)(a - b) = \vee\{\pi(\mu(x)) : f(x) = a - b\} \geq \vee\{\pi(\mu(m + n)) : f(m) = a, f(n) = b\}$$

$$\geq \vee\{\pi(\mu(m) \wedge \mu(n)) : f(m) = a, f(n) = b\}$$

$$= \wedge[\vee\{\pi(\mu(m)) : f(m) = a\}, \vee\{\pi(\mu(n)) : f(n)) = b\}] = \wedge\{f_\pi(\mu)(a), f_\pi(\mu)(b)\}$$

$$f_\pi(\mu)(ab) = \vee\{\pi(\mu(x)) : f(x) = ab\} \geq \vee\{\pi(\mu(mn)) : f(mn) = ab\}$$

$$\geq \vee\{\pi(\mu(m)) : f(m) = a\} = f_\pi(\mu)(a)$$

Similarly it can be proved that $f_\pi(\mu)(ab) \geq f_\pi(\mu)(b)$. Hence $f_\pi(\mu)$ is fuzzy invariant sub near ring of K.

Note 3. The above result also holds if π is an increasing mapping.

Theorem 3.7. Let mapping $f_\pi : \rho(N) \to \rho(K)$ where N, K are near rings, is a fuzzy homomorphism and θ be a fuzzy invariant sub near ring (ideal) of K. Then $f_\pi^{-1}(\theta)$ is fuzzy invariant sub near ring (ideal) of N if f_π is also onto which is constant on ker(f).

Proof. Let x, y $\in$ N. If $f(x), f(y) \in \theta^t$ then $f(x) + f(y) = f(x + y) \in \theta^t$ and we have

$$f_\pi^{-1}(\theta)(x + y) = \pi^{-1}[\theta(f(x + y))] \geq \pi^{-1}[\theta(f(x)) \wedge \theta(f(y))] = \pi^{-1}[\theta(f(x))] \wedge \pi^{-1}[\theta(f(y))]$$

$$= f_\pi^{-1}(\theta)(x) \wedge f_\pi^{-1}(\theta)(y)$$

It can be easily proved that $f_\pi^{-1}(\theta)(-x) \geq f_\pi^{-1}(\theta)(x)$, $f_\pi^{-1}(\theta)(xy) \geq f_\pi^{-1}(\theta)(x)$ and $f_\pi^{-1}(\theta)(xy) \geq f_\pi^{-1}(\theta)(y)$.

Next let $x \in$ ker(f). Then $f(x) \in \theta^t$ and therefore $f_\pi^{-1}(\theta)(x) = \pi^{-1}[\theta(f(x)] = \pi^{-1}[\theta(f(0))] = \pi^{-1}[\theta(0)]$. Hence $f_\pi^{-1}(\theta)$ is constant on ker(f).

Theorem 3.8. Let A be an ideal of N and μ be a fuzzy invariant sub near ring of N. Let $\pi : (0, 1] \to (0, 1]$ be an increasing function. A fuzzy subset μ/A of N/A is defined as

$$\mu/A(x + A) = \vee \{\pi(\mu(y)) : y \in x + A\}$$

Then μ/A is a fuzzy invariant sub near ring of N/A.

Proof. We consider the natural epimorphism $f : N \to N/A$ defined by $f(x) = x + A$, for all $x \in N$. By theorem 3.8, $f_\pi(\mu)$ is fuzzy invariant sub near ring of N/A. Now for any $x + A \in N/A$, $f_\pi(\mu)(x + A) = \vee\{\pi(\mu(y)) : f(y) = x + A\} = \vee\{\pi(\mu(y)) : y + A = x + A\}$.

$= \vee\{\pi(\mu(y)) : y \in x + A\} = \mu/A(x + A)$

Thus $f_\pi(\mu) = \mu/A$ and consequently μ/A is fuzzy invariant sub near ring of N/A.

Note 4. μ/A is called π-fuzzy quotient near ring of μ with respect of A.

Theorem 3.9. Let N and K be two near rings. And μ and θ be two fuzzy left N-subgroups of N and K respectively. If $f_\pi : \rho(\mu) \to \rho(\theta)$ is a fuzzy homomorphism and $f : \mu^* \to \theta^*$ is the ordinary homomorphism with $A = \ker(f)$ then

(i) $\mu^*/A = (\mu/A)^*$

(ii) $f(\mu^*) = (f_\pi(\mu)^*$

Proof. (i) Let $x + A \in \mu^*/A$. Then $x \in \mu^+$ and hence $\mu(x) > 0$. Now,

$\mu/A(x + A) = \vee\{\pi(\mu(y)): y \in x + A\} = \vee\{\pi(\mu(y)) : y - x \in A\} = \vee\{\pi(\mu(y)):f(y)=f(x)\}$

$= f_\pi(\mu)(f(r)) \geq \pi(\mu(x)) > 0$. Hence $x + A \in (\mu/A)^*$.

Next, let $x - A \in (\mu/A)^*$. Thus $\mu/A(x + A) > 0$. Therefore $f_\pi(\mu)(f(x)) > 0$. It follows that there exists $y \in \mu^+$ such that $f(y) = f(x)$ and $\pi(\mu(x)) > 0$.

Hence $y + A = x + A \in \mu^*/A$.

(ii) Let $f(x) \in f(\mu^*)$. Then $\mu(x) > 0$. Now $f_\pi(\mu)(f(x)) = \vee\{\pi(\mu(y)) : f(y) = f(x)\}$

$\geq \pi(\mu(x)) > 0$. This gives $f(x) \in (f_\pi(\mu))^*$.

Let $f(x) \in (f_\pi(\mu))^+$. Then $f_\pi(\mu)(f(x)) > 0$ and therefore there exists $y \in \mu^*$ such that $f(y) = f(x)$ and $\pi(\mu(x)_ > 0$. Thus $f(y) = f(x) \in f(\mu^*)$. This completes the proof.

Theorem 3.10. Let N and K be two near rings. Let μ and θ be two fuzzy left N-subgroups of N and K respectively. Let f_π: $\rho(\mu) \to \rho(\theta)$ is a fuzzy homomorphism and $A = \ker(f)$. Then $f_\pi(\mu) \cong \mu/A$.

Proof. Since $f : \mu^* \to \theta^*$ be the ordinary near ring homomorphism with $A = \ker(f)$, by ordinary near ring theory, A is an ideal of μ^* and also $\mu^*/A \cong f(\mu^*$. This implies $\mu^*/A \cong (f_\pi(\mu))^*$. Therefore there exists an ordinary isomorphism $\eta : \mu^*/A \to (f_\pi(\mu))^*$ where $\mu^*/A = (\mu/A)^*$. Now if $\pi : (0, 1] \to (0, 1]$ is identity mapping, then $\pi : (0, \mu/A(A)] \to (0, f_\pi(\mu)(\mathbf{0})]$ is strictly increasing onto mapping where $\mu/A(A) = f_\pi(\mu)(\mathbf{0})$. Thus $\eta_\pi : \rho(\mu/A) \to \rho(f_\pi(\mu))$ is a fuzzy isomorphism and hence $f_\pi(\mu) \cong \mu/A$.

Theorem 3.11. Let N and K be two near rings and mapping $f_\pi : \rho(N) \to \rho(K)$ is a fuzzy homomorphism where π is identity mapping. Let μ and θ be two fuzzy ideals of N and K respectively. Then

(i) $f_\pi(\mu)(f(x)) = \mu(x)$, for all $x \in N$ if μ is constant on ker(f).

(ii) $(f_\pi^{-1} o f_\pi)(\mu) = \mu$ if μ is constant on ker(f).

(iii) $(f_\pi \text{ o } f_\pi^{-1})(\theta) = \theta$ if f is onto.

Proof. (ii) Let $x \in N$. Let $f(x) \in (f_\pi(\mu))^*$. Then,

$(f_\pi^{-1} {}^\circ f_\pi)(\mu)(x) = f_\pi^{-1}\{f_\pi(\mu)(x)\} = f_\pi(\mu)(f(x)) = \mu(x)$, by (i).

If $f(x) \notin (f_\pi(\mu))^*$, by definition, $(f_\pi^{-1} {}^\circ f_\pi)(\mu)(x) = f_\pi^{-1}\{f_\pi(\mu)(x)\} = 0$.

Also, $f_\pi(\mu)(f(x)) = 0$ implies $\vee\{\pi(\mu(y)) : f(y) = f(x)\} = 0$ Thus $\mu(x) = 0$.

and the result follows.

(iii) Let y be any element of K. Then there exists $x \in N$ such that $f(x) = y$. Now,

$(f_\pi \circ f_\pi^{-1})(\theta)(y) = f_\pi[f_\pi^{-1}(\theta)(y)] = f_\pi[f_\pi^{-1}(\theta)(f(x))] = [f_\pi^{-1}(\theta)](x)$

$= \theta(y)$, if $f(x) \in \theta^*$.

Also if $f(x) = y \notin \theta^*$ then $(f_\pi \circ f_\pi^{-1})(\theta)(y) = 0 = \theta(y)$. Therefore $(f_\pi \text{ o } f_\pi^{-1})(\theta) = \theta$.

Theorem 3.12. Let N and K be two near rings and mapping $f_\pi : \rho(N) \to \rho(K)$ is a fuzzy epimorphism where π is identity mapping. Then there is a one to one order preserving corresponding between the fuzzy ideals of K and the fuzzy ideals of N which are constant on ker(f).

Proof. Let X be the set of all fuzzy ideals of N which are constant on ker(f) and Y be the set of all fuzzy ideals of K. Define $\alpha : X \to Y$ by $\alpha(\mu) = f_\pi(\mu)$ and $\beta : Y \to X$ by $\beta(\theta) = f_\pi^{-1}(\theta)$. Then α and β are well defined and inverses of each other, by theorem 3.11.(ii), (iii). Thus we get the required order preserving one to one correspondence between X and Y.

5. EXAMPLE

We consider the set N = {0, a, b, c} and binary operations +, ., 0 as follows

+	0	a	b	c
0	0	a	b	c
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
b	0	b	0	0
c	0	0	b	c

o	0	a	b	c
0	0	0	0	0
b	0	0	b	b
c	0	a	0	c

Then it is seen that (N, +, .) and (N, +, o) are two near-rings. We define a fuzzy subset μ of (N, +, .) and θ of (N, +, o) as follows:

$\mu(0) = \mu(b) = 1$, $\mu(a) = \mu(c) = s$ where $s \in (0,1)$

and $\theta(0) = \theta(a) = 1$, $\theta(b) = \theta(c) = t$, $t \in (0, 1)$. Then μ is a fuzzy left N-subgroup of (N, +, .) and θ is a fuzzy left N-subgroup of. (N, +, o).

We define a mapping $F : \rho(\mu) \to \rho(\theta)$ such that $F(0_u) = 0_u = F(b_u)$,

$F(a_u) = a_u = F(c_u)$, for all $u \in (0,1]$. Then F is a fuzzy homomorphism. So, there exists an ordinary homomorphism $f : \mu^* \to \theta^*$ where $\mu^* = \{0, a, b, c\} = \theta^*$ such that $f(0) = 0 = f(b)$, $f(a) = f(c) = a$ and an increasing function $\pi : (0,1] \to (0,1]$ such that $\pi(t) = t$. It is clear that $F(x_t) = [F(x)]_{\pi(t)}$.

Conclusion: The notions of fuzzy homomorphism and fuzzy isomorphism may lead to the fuzzy concept of exact sequence of N-homomorphism and using this concept one may define projective fuzzy N-group and injective fuzzy N-group.

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Jour. Pure Math., Vol. 23, 2006, pp. 65-80

SOME GENERATING FUNCTIONS DUE TO CERTAIN ENTRIRE FUNCTIONS

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ABSTRACT : In the present paper, we introduce an entire function by the product of n-different variables Mittag-Leffler's functions E_α and $E_{\alpha,\beta}$. Then, we derive its related theorems and make their applications to obtain many special known and unknown generating functions.

Key words and pharases : Mittag-Leffler functions, Hyper-Bessel functions, Generating functions, Multivariable generalized Bessel function, Generalized Lauricella series in several variable due to Srivastava and Daoust, Multivariable generalized Kampe'de Fe'riet function due to Srivastava and Panda
2000 AMS Classification : 33 C 99, 33 E 12.

1. INTRODUCTION

Mittag – Leffler [5, 6] defined an entire function of order $1/\alpha$ as

$$E_\alpha(z) = \sum_{s=0}^{\infty} \frac{(z)^s}{\Gamma(\alpha s + q)} \qquad \text{..... (1.1)}$$

provided that Re $(\alpha) > 0$ and $z \in \mathbb{C}$. That is called a Mittag - Leffler function.

A generalization of (1.1) in the form is given by

$$E_{\alpha,\beta}(z) = (z)^{(\beta-1)/\alpha} \sum_{s=0}^{\infty} \frac{(z)^s}{\Gamma(\alpha s + \beta)} \qquad \text{..... (1.2)}$$

provided that Re $(\alpha) > 0$, Re $(\beta) > 0$ and $z \in \mathbb{C}$. That is also an entire function of order $1/\alpha$ and has similar properties to that of (1.1). (See, Wiman [13,14] and Agrawal [1]).

These functions are studied by many authors including Mittag - Leffler [5, 6], Erde'lyi [2], Humbert [4], and Humbert and Agrawal [5] etcetra. (Also, see Srivastava and Manocha [11]).

During various developments of fractional calculus, these functions have gained importance and popularity due to direct applications in the field of physical and mathematical sciences, engineering and computation theory.

The Gamma function is defined by

$$\Gamma(\lambda + n) = \Gamma(\lambda).(\lambda)_n \qquad \text{.....}\ (1.3)$$

where, $\lambda \neq 0, -1, -2, \ldots\ldots$, and $(\lambda)_n$ is the Pochhammer symbol which is defined by

$$\begin{aligned}(\lambda)_n &= \lambda(\lambda + 1) \ldots\ldots (\lambda + n - 1), && n = 1, 2, 3, \ldots\ldots \\ &= 1, && n = 0\end{aligned} \qquad \text{.....}\ (1.4)$$

Obviously, from (1.2), we have

$$E_{\alpha 1}(z) = E_{\alpha}(z) \qquad \text{.....}\ (1.5)$$

$$E_{1,\,1}(z) = e^{z} \qquad \text{.....}\ (1.6)$$

$$E_{2,\,1}(-z^2) = \cos z \qquad \text{.....}\ (1.7)$$

$$E_{2,\,2}(-z^2) = \sin z \qquad \text{.....}\ (1.8)$$

$$E_{2,\,1}(z^2) = \cos hz \qquad \text{....}\ (1.9)$$

$$E_{2,\,2}(z^2) = \sin hz \qquad \text{.....}\ (1.10)$$

$$F_{1/2,1}(\sqrt{z}) = \frac{2e^{-z}}{\sqrt{\pi}} \mathrm{Erfc}(-\sqrt{z}) \qquad \text{....}\ (1.11)$$

Following functions, occured in the various fields of physics and engineering, are recalled here, in our work, in the form :

$$J_{\nu}^{\mu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{s=0}^{\infty} \frac{(-z^2/4)^s}{s!\,\Gamma(\mu(s+\nu)+1)} \qquad \text{.....}\ (1.12)$$

which is the generalized Bessel function.

$$J_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\ {}_0F_2[-; m+1, n+1; -(z/3)^3] \qquad \text{....}\ (1.13)$$

which is the hyper - Bessel function.

$$I_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\ {}_0F_2[-; m+1, n+1; (z/3)^3] \qquad \text{....}\ (1.14)$$

which is the modified hyper-Bessel function.

The function ${}_pF_q$ [.] is the generalized hypergeometric function of one variable, with p numerator parameters and q denominator parameters (non negative and non zero) defined by Rainville [8].

Motivated by above work and by the well known property of entire functions that product of entire functions is also an entire function, an entire function that is defined by

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}[\lambda z_1 x_1,\ldots,\lambda z_n x_n]$$

$$= (\lambda z_1 x_1)^{\left(\frac{(\beta_1-1)}{\alpha_1}\right)} \ldots\ldots (\lambda z_n x_n)^{\left(\frac{\beta_n-1}{\alpha_n}\right)} \sum_{s_1,\ldots,s_n=0}^{\infty} \frac{(\lambda z_1 x_1)^{s_1}\ldots(\lambda z_n x_n)^{s_n}}{\Gamma(\alpha_1 s_1+\beta_1)\ldots\ldots\Gamma(\alpha_n s_n+\beta_n)} \quad \ldots\ldots (1.15)$$

provided that all Re $(\alpha_i) > 0$, Re $(\beta_i) > 0$, all z_i be finite and all x_i are not equal to zero where, $i = 1, \ldots, n$. λ may be real or complex.

Particularly,

$$E_{1,1,\ldots,1,1}[\lambda z_1 x_1,\ldots,\lambda z_n x_n] = \exp\left[\lambda \sum_{i=1}^{n} z_i x_i\right] \quad \ldots\ldots (1.15a)$$

So that an entire function defined by (1.15) is the product of n-different variables Mittg-Lefler functions.

Now, making an use of (1.2) and (1.15), we get

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}[\lambda z_1 x_1,\ldots\ldots\lambda z_n x_n] = \prod_{i=1}^{n} E_{\alpha_i,\beta_i}[\lambda z_i x_i] \quad \ldots\ldots (1.16)$$

And then, setting $z_1 = z\,x_1$, ..., $z_n = z\,x_n$ and $\lambda = -z$ in (1.16) both sides and with the aid of the results (1.7) and (1.8), we get

$$E_{2,1,\ldots,2,1}[-z^2x_1^2,\ldots\ldots,-z^2x_n^2] = \prod_{i=1}^{n} \cos zx_i \quad \ldots\ldots (1.17)$$

and

$$E_{2,2,\ldots,2,2}[-z^2x_1^2,\ldots\ldots,-z^2x_n^2] = \prod_{i=1}^{n} \sin zx_i \quad \ldots\ldots (1.18)$$

respectively.

Again, setting $z_1 = z\,x_1$, ..., $z_n = zx_n$ and $\lambda = z$ in (1.16) both sides, and then using the results (1.9) and (1.10), we get

$$E_{2,1,\ldots,2,1}[z^2x_1^2,\ldots\ldots,z^2x_n^2] = \prod_{i=1}^{n} \cos hzx_i \quad \ldots\ldots (1.19)$$

and $$E_{2,2,\ldots,2,2}[z^2x_1^2,\ldots\ldots,z^2x_n^2] = \prod_{i=1}^{n} \sin hzx_i \quad \ldots\ldots (1.20)$$

respectively

Further, setting $x_1 = z_1^{-1/2}, \ldots\ldots, x_n = z_n^{-1/2}$ and $\lambda = 1$ in (1.16) both sides and then using the result (1.11), we find

$$E_{1/2,1\;\;,1/2,1}[\sqrt{z_1},\ldots,\sqrt{z_n}] = \left(\frac{2}{\sqrt{\pi}}\right)^n \exp[-(z_1+\ldots+z_n)] \prod_{i=1}^{n} \mathrm{Erfc}(-\sqrt{z_i}) \qquad \ldots\ldots(1.21)$$

And futher, setting $z_1 = zx_1^{-1/2}, \ldots\ldots, z_n = zx_n^{-1/2}$ and $\lambda = z^{-1/2}$ in (1.16) both sides and then using the result (1.11), we find

$$E_{1/2,1\;\;,1/2,1}[\sqrt{zx_1},\ldots,\sqrt{zx_n}] = \left(\frac{2}{\sqrt{\pi}}\right)^n \exp[-(zx_1+\ldots\ldots+zx_n)] \prod_{i=1}^{n} \mathrm{Erfc}(-\sqrt{zx_i}) \qquad \ldots\ldots(1.22)$$

Again, in (1.15) for $n = 3$, setting $\lambda = 1/3$, $z_1 = z_2 = z_3 = z$, $x_1 = x$, $x_2 = y$ and $x_3 = -1/xy$ and then, making an use of series rearrangement techniques, we find

$$E_{\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3}\left[\frac{zx}{3},\frac{zy}{3},\frac{-z}{3xy}\right]$$

$$= \left(\frac{z}{3}\right)^{\sum_{i=1}^{3}\left(\frac{\beta_i-1}{\alpha_i}\right)} (x)^{\left(\frac{\beta_1-1}{\alpha_1}-\frac{\beta_3-1}{\alpha_3}\right)} (y)^{\left(\frac{\beta_2-1}{\alpha_2}-\frac{\beta_3-1}{\alpha_3}\right)} (-1)^{\frac{\beta_3-1}{\alpha_3}}$$

$$\cdot \sum_{m,n=-\infty}^{\infty} x^m y^n \, {}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3} J_{m,n}(z) \qquad \ldots\ldots(1.23)$$

where

$${}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3} J_{m,n}(z) = \left(\frac{z}{3}\right)^{m+n} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z}{3}\right)^{3s}}{\Gamma(\alpha_1(m+s)+\beta_1)\,\Gamma(\alpha_2(n+s)+\beta_2)} \frac{1}{\Gamma(\alpha_3 s+\beta_3)} \qquad \ldots\ldots(1.24)$$

which may be a generalized hyper – Bessel function.

Further, in (1.15) for $n = 3$, setting $\lambda = 1/3$, $z_1 = z_2 = z_3 = z$, $x_1 = x$, $x_2 = y$ and $x_3 = 1/xy$, and then, making an use of series rearrangement techniques, we get

$$E_{\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3}\left[\frac{zx}{3},\frac{zy}{3},\frac{z}{3xy}\right] = \left(\frac{z}{3}\right)^{\sum_{i=1}^{3}\left(\frac{\beta_i-1}{\alpha_i}\right)}(x)^{\left(\frac{\beta_1-1}{\alpha_1}-\frac{\beta_3-1}{\alpha_3}\right)}(y)^{\left(\frac{\beta_2-1}{\alpha_2}-\frac{\beta_3-1}{\alpha_3}\right)}$$

$$\times \sum_{m,n=-\infty}^{\infty} x^m y^n \cdot {}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}I_{m,n}(z) \qquad \ldots\ldots (1.25)$$

where

$${}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}I_{m,n}(z) = \left(\frac{z}{3}\right)^{m+n}\sum_{s=0}^{\infty}\frac{\left(\frac{z}{3}\right)^{3s}}{\Gamma(\alpha_1(m+s)+\beta_1)\Gamma(\alpha_2(n+s)+\beta_2)\Gamma(\alpha_3 s+\beta_3)} \qquad \ldots\ldots (1.26)$$

which may be a generalized modified hyper-Bessel function.

Obviously, from (1.13) and (1.24), we have

$${}^{1,1,1}_{1,1,1}J_{m,n}(z) = J_{m,n}(z) \qquad \ldots\ldots (1.27)$$

and from (1.14) and (1.26), we have

$${}^{1,1,1}_{1,1,1}I_{m,n}(z) = I_{m,n}(z) \qquad \ldots\ldots (1.28)$$

Now, setting $\lambda = \frac{1}{n+1}$, $z_1 = \ldots = z_n = z_{n+1} = z$ and $x_{n+1} = -\frac{1}{x_1\ldots\ldots x_n}$ in (1.15) for $n = n+1$, we get

$$E_{\alpha_1,\beta_1,\ldots\ldots\alpha_n,\beta_n,\alpha_{n+1},\beta_{n+1}}\left[\frac{zx_1}{n+1},\ldots,\frac{zx_n}{n+1},\frac{-z}{(n+1)(x_1\ldots x_n)}\right]$$

$$= \left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n+1}\left(\frac{\beta_i-1}{\alpha_i}\right)} \cdot \prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\frac{\beta_{n+1}-1}{\alpha_{n+1}}\right)}(-1)^{\left(\frac{\beta_{n+1}-1}{\alpha_{n+1}}\right)}$$

$$\times \sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots\ldots x_n^{m_n} \cdot {}^{\alpha_1,\ldots,\alpha_n,\alpha_{n+1}}_{\beta_1,\ldots,\beta_n,\beta_{n+1}}J_{m_1,\ldots,m_n}(z) \qquad \ldots\ldots (1.29)$$

where,

$$
{}^{\alpha_1,\ldots,\alpha_n,\alpha_{n+1}}_{\beta_1,\ldots,\beta_n,\beta_{n+1}} J_{m_1,\ldots,m_n}(z) = \left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n} m_i} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\left(\frac{z}{n+1}\right)^{n+1}\right)^s}{\Gamma(\alpha_1(m_1+s)+\beta_1)\ldots}
$$

$$
\times \frac{1}{\ldots\Gamma(\alpha_n(m_n+s)+\beta_n)\Gamma(\alpha_{n+1}s+\beta_{n+1})} \qquad \ldots\ldots (1.30)
$$

Further setting $\lambda = \frac{1}{n+1}$, $z_1 = \ldots\ldots = z_n = z_{n+1} = z$ and $x_{n+1} = \frac{1}{x_1\ldots\ldots x_n}$ in (1.15) for $n = n+1$, we find

$$
E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n,\alpha_{n+1},\beta_{n+1}}\left[\frac{zx_1}{n+1},\ldots\ldots,\frac{zx_n}{n+1},\frac{z}{(n+1)(x_1\ldots\ldots x_n)}\right]
$$

$$
= \left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n+1}\left(\frac{\beta_i-1}{\alpha_i}\right)} \cdot \prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\frac{\beta_{n+1}-1}{\alpha_{n+1}}\right)}
$$

$$
\times \sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots\ldots\lambda_n^{m_n} \cdot {}^{\alpha_1,\ldots\ldots,\alpha_n,\alpha_{n+1}}_{\beta_1,\ldots\ldots,\beta_n,\beta_{n+1}} I_{m_1,\ldots,m_n}(z) \qquad \ldots\ldots (1.31)
$$

where,

$$
{}^{\alpha_1,\ldots\ldots,\alpha_n,\alpha_{n+1}}_{\beta_1,\ldots\ldots,\beta_n,\beta_{n+1}} I_{m_1,\ldots,m_n}(z) = \left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n} m_i} \sum_{s=0}^{\infty} \frac{\left(\left(\frac{z}{n+1}\right)^{n+1}\right)^s}{\Gamma(\alpha_1(m_1+s)+\beta_1)\ldots\ldots\Gamma(\alpha_n(m_n+s)+\beta_n)}
$$

$$
\times \frac{1}{\Gamma(\alpha_{n+1}s+\beta_{n+1})} \qquad \ldots\ldots (1.32)
$$

2. SOME THEOREMS RELATED TO THE ENTIRE FUNCTIONS

Theorem 1. If all Re $(\alpha_i) > 0$, Re $(\beta_i) > 0$, Re $(\gamma_i) > 0$, Re $(\delta_i) > 0$ and all $x_i \neq 0$, where, $i = 1, \ldots, n$, all $z_1 = \ldots = z_n = z$ be finite and $\lambda = 1/2$, then by the definition (1.15),

we have

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}\left[z\frac{x_1}{2},\ldots\ldots,z\frac{x_n}{2}\right]E_{\gamma_1,\delta_1,\ldots,\gamma_n,\delta_n}\left[\frac{-z}{2x_1},\ldots\ldots,\frac{-z}{2x_n}\right]$$

$$=\left(\frac{z}{2}\right)^{\sum\limits_{i=1}^{n}\left(\frac{\beta_i-1}{\alpha_i}+\frac{\delta_i-1}{\gamma_i}\right)}.(-1)^{\sum\limits_{i=1}^{n}\left(\frac{\delta_i-1}{\gamma_i}\right)}$$

$$\prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\frac{\delta_i-1}{\gamma_i}\right)}.\sum_{m_1,\ldots,m_n=-\infty}^{\infty}x_1^{m_1}\ldots\ldots x_n^{m_n}\times\prod_{i=1}^{n}\left(\begin{matrix}\alpha_i,\gamma_i\\ \beta_i,\delta_i\end{matrix}J_{m_i}(z)\right) \qquad \ldots\ldots (2.1)$$

where

$$\begin{matrix}\alpha_i,\gamma_i\\ \beta_i,\delta_i\end{matrix}J_{m_i}(z)=\left(\frac{z}{2}\right)^{m_i}\sum_{t_i=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^{t_i}}{\Gamma(\alpha_i(m_i+t_i)+\beta_i)\Gamma(\gamma_i t_i+\delta_i)}$$

$i = 1, \ldots\ldots, n.$ ..... (2.2)

Proof. Recalling the definition (1.15), we have

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}\left[\frac{zx_1}{2},\ldots\ldots,\frac{zx_n}{2}\right]\cdot E_{\gamma_1,\delta_1,\ldots,\gamma_n,\delta_n}\left[\frac{-z}{2x_1},\ldots\ldots,\frac{-z}{2x_n}\right]$$

$$=\left(\frac{z}{2}\right)^{\sum\limits_{i=1}^{n}\left(\frac{\beta_i-1}{\alpha_i}+\frac{\delta_i-1}{\gamma_i}\right)}\prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\frac{\delta_i-1}{\gamma_i}\right)}(-1)^{\sum\limits_{i=1}^{n}\left(\frac{\delta_i-1}{\gamma_i}\right)}$$

$$\times\sum_{s_1,\ldots,s_n=0}^{\infty}\;\sum_{t_1,\ldots,t_n=0}^{\infty}\frac{(-1)^{t_1+\ldots+t_n}(z/2)^{s_1+t_1+\ldots+s_n+t_n}x_1^{s_1-t_1}\ldots\ldots x_n^{s_n-t_n}}{\Gamma(\alpha_1 s_1+\beta_1)\ldots\ldots\Gamma(\alpha_n s_n+\beta_n)\Gamma(\gamma_1 t_1+\delta_1)\ldots\ldots\Gamma(\gamma_n t_n+\delta_n)} \qquad \ldots\ldots (2.3)$$

Now, setting $s_1 - t_1 = m_1, \ldots\ldots, s_n - t_n = m_n$ in the right hand side of (2.3), and then making an use of series rearrangement techniques, we find

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}\left[\frac{zx_1}{2},\ldots\ldots,\frac{zx_n}{2}\right]\cdot E_{\gamma_1,\delta_1,\ldots,\gamma_n,\delta_n}\left[\frac{-z}{2x_1},\ldots\ldots,\frac{-z}{2x_n}\right]$$

$$=\left(\frac{z}{2}\right)^{\sum\limits_{i=1}^{n}\left(\frac{\beta_i-1}{\alpha_i}+\frac{\delta_i-1}{\gamma_i}\right)}\prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\frac{\delta_i-1}{\gamma_i}\right)}\cdot(-1)^{\sum\limits_{i=1}^{n}\left(\frac{\delta_i-1}{\gamma_i}\right)}$$

$$\cdot \sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1} \ldots\ldots \lambda_n^{m_n} \left({}^{\alpha_1,\gamma_1}_{\beta_1,\delta_1} J_{m_1}(z) \ldots\ldots {}^{\alpha_n,\gamma_n}_{\beta_n,\delta_n} J_{m_n}(z) \right) \quad \ldots\ldots (2.4)$$

where

$${}^{\alpha_1,\gamma_1}_{\beta_1,\delta_1} J_{m_1}(z) = \left(\frac{z}{2}\right)^{m_1} \sum_{t_1=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^{t_1}}{\Gamma(\alpha_1(m_1+t_1)+\beta_1)\Gamma(\gamma_1 t_1+\delta_1)}$$

....

$${}^{\alpha_n,\gamma_n}_{\beta_n,\delta_n} J_{m_n}(z) = \left(\frac{z}{2}\right)^{m_n} \sum_{t_n=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^{t_n}}{\Gamma(\alpha_n(m_n+t_n)+\beta_n)\Gamma(\gamma_n t_n+\delta_n)} \quad \ldots\ldots (2.5)$$

which are generalized Bessel functions analogous to (1.12). Hence, due to (2.4) and (2.5), we find (2.1) and (2.2) respectively. Thus, above theorem has been proved.

Theorem-2. If all $Re(\alpha_i) > 0$, $Re(\beta_i) > 0$, RE $(\gamma_i) > 0$, Re $(\delta_i) > 0$ and all $x_i \neq 0$, where, $i = 1, \ldots, n$, all $z_1 = \ldots = z_n = z$ be finite and $\lambda = \frac{1}{2}$. then by (1.15), we have

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}\left[\frac{z\lambda_1}{2}, \ldots\ldots, \frac{zx_n}{2}\right] \cdot E_{\gamma_1,\delta_1,\ldots,\gamma_n,\delta_n}\left[\frac{z}{2x_1}, \ldots\ldots, \frac{z}{2x_n}\right]$$

$$= \left(\frac{z}{2}\right)^{\sum_{i=1}^{n}\left(\frac{\beta_i-1}{\alpha_i}+\frac{\delta_i-1}{\gamma_i}\right)} \prod_{i=1}^{n} (x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\frac{\delta_i-1}{\gamma_i}\right)} \times \sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1} \ldots\ldots x_n^{m_n} \prod_{i=1}^{n} \left({}^{\alpha_i,\gamma_i}_{\beta_i,\delta_i} I_{m_i}(z) \right) \quad \ldots\ldots (2.6)$$

where,

$${}^{\alpha_i,\gamma_i}_{\beta_i,\delta_i} I_{m_i}(z) = \left(\frac{z}{2}\right)^{m_i} \sum_{t_i=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^{t_i}}{\Gamma(\alpha_i(m_i+t_i)+\beta_i)\Gamma(\gamma_i t_i+\delta_i)}$$

$i = 1, \ldots\ldots, n.$ ..... (2.7)

which are generalized modified Bessel functions.

Proof. Making an appeal to the techniques applied to prove theorem 1, we can prove theorem 2 in similar fashion.

Theorem 3. If all $\mathrm{Re}(\alpha_i) > 0$, $\mathrm{Re}\,(\beta_i) > 0$, $\mathrm{Re}\,(\gamma_i) > 0$, $\mathrm{Re}(\delta_i) > 0$ and $x_i \neq 0$ where, $i = 1, \ldots..n$, all $z_1 = \ldots. - z_n = z$ be finite and $\lambda = \frac{1}{n+1}$, then by (1.15), we have

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}\left[\frac{zx_1}{n+1},\ldots..,\frac{zx_n}{n+1}\right]\cdot E_{\gamma_1,\delta_1,\ldots,\gamma_n,\delta_n}\left[\frac{-z}{(n+1)(x_1\ldots x_n)},\ldots..,\frac{-z}{(n+1)(x_1\ldots..x_n)}\right]$$

$$= \left(\frac{z}{n+1}\right)^{\sum\limits_{i=1}^{n}\left(\frac{\beta_i-1}{\alpha_i}+\frac{\delta_i-1}{\gamma_i}\right)}\prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\sum\limits_{j=1}^{n}\left(\frac{\delta_j-1}{\gamma_j}\right)\right)}\times(-1)^{\sum\limits_{i=1}^{n}\left(\frac{\delta_i-1}{\gamma_i}\right)} \qquad \ldots..\ (2.8)$$

$$\cdot\sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots..x_n^{m_n}\cdot {}^{\alpha_1,\ldots..,\alpha_n;\gamma_1,\ldots..,\gamma_n}_{\beta_1,\ldots..,\beta_n;\delta_1,\ldots..,\delta_n}J_{m_1,\ldots,m_n}[z,\ldots..z]$$

where

$${}^{\alpha_1,\ldots...,\alpha_n;\gamma_1,\ldots..,\gamma_n}_{\beta_1,\ldots..,\beta_n;\delta_1,\ldots...,\delta_n}J_{m_1,\ldots,m_n}[z,\ldots..z] = \left(\frac{z}{n+1}\right)^{m_1+\ldots+m_n}$$

$$\cdot\sum_{t_1,\ldots,t_n=0}^{\infty}\frac{\left(-\left(\frac{z}{n+1}\right)^{n+1}\right)^{t_1}\ldots..\left(-\left(\frac{z}{n+1}\right)^{n+1}\right)^{t_n}}{\Gamma\left(\alpha_1(m_1+t_1+\ldots.+t_n)+\beta_n\right)\ldots..}$$

$$\times\frac{1}{\ldots..\Gamma\left(\alpha_n(m_n+t_1+\ldots.+t_n)+\beta_n\right)\Gamma(\gamma_1t_1+\delta_1)\ldots..\Gamma(\gamma_nt_n+\delta_n)} \qquad \ldots..\ (2.9)$$

which may be multivariable generalized Bessel function.

Proof. Recalling the definition (1.15), we have

$$E_{\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}\left[\frac{zx_1}{n+1},\ldots,\frac{zx_n}{n+1}\right]E_{\gamma_1,\delta_1,\ldots,\gamma_n,\delta_n}\left[\frac{-z}{(n+1)(x_1\ldots x_n)},\ldots,\frac{-z}{(n+1)(x_1\ldots x_n)}\right]$$

$$= \left(\frac{z}{n+1}\right)^{\sum\limits_{i=1}^{n}\left(\frac{\beta_i-1}{\alpha_i}+\frac{\delta_i-1}{\gamma_i}\right)}\prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\sum\limits_{j=1}^{n}\left(\frac{\delta_j-1}{\gamma_j}\right)\right)}$$

$$\times(-1)^{\sum\limits_{i=1}^{n}\left(\frac{\delta_i-1}{\gamma_i}\right)}\sum_{s_1,\ldots,s_n=0}^{\infty}\ \sum_{t_1,\ldots,t_n=0}^{\infty}\frac{(-1)^{t_1+\ldots+t_n}}{\Gamma(\alpha_1s_1+\beta_1)\ldots\Gamma(\alpha_ns_n+\beta_n)}$$

$$\times\left(\frac{z}{n+1}\right)^{s_1+\ldots+s_n+t_1+\ldots t_n}\frac{(x_1)^{s_1-t_1\ldots-t_n}\ldots(x_n)^{s_n-t_1\ldots-t_n}}{\Gamma(\gamma_1t_1+\delta_1)\ldots..\Gamma(\gamma_nt_n+\delta_n)}$$

now, in right hand side of above, setting $s_1 - t_1 - \ldots\ldots - t_n = m_1, \ldots\ldots s_n - t_1 - \ldots\ldots - t_n = m_n$ and then using series rearrangement techniques, we get

$$= \left(\frac{z}{n+1}\right)^{\sum\limits_{i=1}^{n}\left(\frac{\beta_i - 1}{\alpha_i} + \frac{\delta_i - 1}{\gamma_i}\right)} \prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i - 1}{\alpha_i} - \sum\limits_{j=1}^{n}\left(\frac{\delta_j - 1}{\gamma_j}\right)\right)} (-1)^{\sum\limits_{i=1}^{\infty}\left(\frac{\delta_i - 1}{\gamma_i}\right)}$$

$$\sum_{m_1, \ldots, m_n = -\infty}^{\infty} x_1^{m_1} \ldots x_n^{m_n} \left(\frac{z}{n+1}\right)^{m_1 + \ldots + m_n} \sum_{t_1 \ldots t_n = 0}^{\infty} \frac{\left(-\left(\frac{z}{n+1}\right)^{n+1}\right)^{t_1} \ldots\ldots \left(-\left(\frac{z}{n+1}\right)^{n+1}\right)^{t_n}}{\Gamma\left(\alpha_1(m_1 + t_1 + \ldots + t_n) + \beta_1\right) \ldots\ldots}$$

$$\times \frac{1}{\ldots\ldots\Gamma\left(\alpha_n(m_n + t_1 + \ldots + t_n) + \beta_n\right)\Gamma\left(\alpha_1 t_1 + \delta_1\right) \ldots \Gamma\left(\gamma_n t_n + \delta_n\right)}$$

Hence, the theorem 3 is proved.

Particularly, from (2.9), and with the aid of the well – known formula

$$\sum_{m_1, \ldots, m_n = 0}^{\infty} f(m_1 + \ldots\ldots + m_n) \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!} = \sum_{m=0}^{\infty} f(m) \frac{(x_1 + \ldots + x_n)^m}{m!}$$

we find

$$\begin{matrix} \alpha_1, \ldots\ldots, \alpha_n;\ 1, \ldots\ldots, 1 \\ \beta_1, \ldots\ldots, \beta_n;\ 1, \ldots\ldots, 1 \end{matrix} J_{m_1, \ldots, m_n}[z, \ldots, z] = \left(\frac{z}{n+1}\right)^{m_1 + \ldots + m_n}$$

$$\times \sum_{t=0}^{\infty} \frac{\left(-n\left(\frac{z}{n+1}\right)^{n+1}\right)^t}{\Gamma\left(\alpha_1(m_1 + t) + \beta_1\right) \ldots\ldots \Gamma\left(\alpha_n(m_n + t) + \beta_n\right) t!} \qquad \ldots\ldots (2.10)$$

which is a generalization of the function the due to Hayeck and Suarez [3], defined by

$$C^{(n)}_{\lambda_1, \ldots, \lambda_n}(x) = \sum_{x=0}^{\infty} \frac{(-x)^r}{\Gamma(\lambda_1 + r + 1) \ldots\ldots \Gamma(\lambda_n + r + 1) r!} \qquad \ldots\ldots (2.11)$$

Hence, due to above theorem, other results involving the function $C^{(n)}_{\lambda_1, \ldots, \lambda_n}(x)$ may be evaluated.

Theorem 4. If all $\mathrm{Re}(\alpha_i) > 0$, $\mathrm{Re}(\beta_i) > 0$, $\mathrm{Re}(\gamma_i) > 0$, $\mathrm{Re}(\delta_i) > 0$ and all $x_i \neq 0$, where, $i = 1, ..., n$, all $z_1 = ... z_n = z$ be finite and $\lambda = \frac{1}{n+1}$, then by (1.15), we have

$$E_{\sigma_1,\beta_1,\ ,\gamma_n,\delta_n}\left[\frac{zx_1}{n+1},\dots,\frac{zx_n}{n+1}\right]\cdot E_{\gamma_1,\delta_1,\ ,\gamma_n,\delta_n}\left[\frac{z}{(n+1)(x_1,\dots,x_n)},\dots,\frac{z}{(n+1)(x_1\dots x_n)}\right]$$

$$= \left(\frac{z}{n+1}\right)^{\sum_{i=1}^{n}\left(\frac{\beta_i-1}{\alpha_i}+\frac{\delta_i-1}{\gamma_i}\right)} \prod_{i=1}^{n}(x_i)^{\left(\frac{\beta_i-1}{\alpha_i}-\sum_{j=1}^{n}\left(\frac{\delta_j-1}{\gamma_j}\right)\right)}$$

$$\cdot \sum_{m_1,\ ,m_n=-\infty}^{\infty} x_1^{m_1}\dots x_n^{m_n}\ {}^{\alpha_1,\dots,\alpha_n;\gamma_1,\dots\gamma_n}_{\beta_1,\dots\beta_n;\delta_1,\dots,\delta_n} I_{m_1,\ ,m_n}[z,\dots z] \qquad ..... (2.12)$$

where

$${}^{\alpha_1,\dots,\alpha_n;\gamma_1,\dots\gamma_n}_{\beta_1,\dots\beta_n;\delta_1,\dots,\delta_n} I_{m_1,\ ,m_n}[z,\dots z]$$

$$= \left(\frac{z}{n+1}\right)^{m_1+\ \ +m_n} \sum_{t_1,\ t_n=0}^{\infty} \frac{\left(\left(\frac{z}{n+1}\right)^{n+1}\right)^{t_1}\cdots\cdots\left(\left(\frac{z}{n+1}\right)^{n+1}\right)^{t_n}}{\Gamma(\alpha_1(m_1+t_1+\dots+t_n)+\beta_i)\dots}$$

$$\frac{1}{\dots\Gamma(\alpha_n(m_n+t_1+\dots+t_n)+\beta_n)\Gamma(\gamma_1 t_1+\delta_1)\dots\dots\Gamma(\gamma_n t_n+\delta_n)} \qquad ..... (2.13)$$

which may be a multivariable generalized modified Bessel function.

Proof. Making an appeal to the techniques applied for proving the theorem 3, we can prove theorem 4 in similar manner.

3. APPLICATIONS

In this section, we make the applications of the theorems stated in the section 2 to evaluate various known and unknown generating functions.

Setting $\alpha_1 = ... = \alpha_n = 2 = \gamma_1 = ... = \gamma_n$ and $\beta_1 = ... = \beta_n = 1 = \delta_1 = ... = \delta_n$ in both sides of the theorem 1 and then making an appeal to the relations (1.17) and (1.19), we obtain the generating function

$$\left\{\cosh\sqrt{\frac{zx_1}{2}}\ldots\ldots\cosh\sqrt{\frac{zx_n}{2}}\right\}\left\{\cos\sqrt{\frac{z}{2x_1}}\ldots\cos\sqrt{\frac{z}{2x_n}}\right\}$$

$$= \sum_{m_1,\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots x_n^{m_n}\frac{(\pi)^{\frac{n}{2}}\left(\frac{z}{8}\right)^{m_1+\quad+m_n}}{\Gamma(m_1+1/2)\Gamma(m_1+1)\ldots\ldots\Gamma(m_n+1/2)\Gamma(m_n+1)}$$

$$\times\ {}_0F_3\left[-;m_1+\frac{1}{2},\ m_1+1,\ \frac{1}{2};-\frac{z^2}{64}\right]\ldots\ldots\ {}_0F_3\left[-;m_n+\frac{1}{2},\ m_n+1,\ \frac{1}{2};-\frac{z^2}{64}\right]\ldots\ldots \quad (3.1)$$

Setting $\alpha_1 = \ldots. = \alpha_n = 2 = \beta_1 = \ldots\ \beta_n$ and $\gamma_1 = \ldots = \gamma_n = 2 = \delta_1 = \ldots = \delta_n$ in both sides of the theorem 1 and then making an appeal to the relations (1.18) and (1.20), we evaluate the generating function

$$\left\{\sin h\sqrt{\frac{zx_1}{2}}\ldots\sin h\sqrt{\frac{zx_n}{2}}\right\}\left\{\sin\sqrt{\frac{z}{2x_1}}\ \ldots\sin\sqrt{\frac{z}{2x_n}}\right\}$$

$$= \sum_{m_1\quad .m_n=-\infty}^{\infty} x_1^{m_1}\ldots x_n^{m_n}\frac{(z/4)^n(-\pi)^{n/2}(z/8)^{m_1+..+m_n}}{\Gamma(m_1+3/2)\Gamma(m_1+1)\ldots\Gamma(m_n+3/2)\Gamma(m_n+1)}$$

$$\times\ {}_0F_3\ [-,\ m_1+3/2,\ m_1+1,\ 3/2;\ -z^2/64]\ \ldots..$$

$$\ldots\times\ {}_0F_3\ [-;\ m_n+3/2;\ m_n+1,\ 3/2;\ -z^2/64] \quad \ldots..\ (3.2)$$

Setting $\alpha_1 = \ldots = \alpha_n = 2 = \gamma_1 = \ldots = \gamma_n$ and $\beta_1 = \ldots = \beta_n = 1 = \delta_1 = \ldots = \delta_n$ in both sides of the theorem 2 and then making an appeal to the relation (1.19), we derive

$$\left\{\cosh\sqrt{\frac{zx_1}{2}}\ldots\ldots\cosh\sqrt{\frac{zx_n}{2}}\right\}\left\{\cosh\sqrt{\frac{z}{2x_1}}\ldots\ldots\cosh\sqrt{\frac{z}{2x_n}}\right\}$$

$$= \sum_{m_1,\quad .m_n=-\infty}^{\infty} x_1^{m_1}\ldots\ldots x_n^{m_n}\frac{(\pi)^{n/2}(z/8)^{m_1+.+m_n}}{\Gamma(m_1+1/2)\Gamma(m_1+1)\ldots\ldots\Gamma(m_n+1/2)\Gamma(m_n+1)}$$

$$\times\ {}_0F_3\ [-;\ m_1+1/2,\ m_1+1,\ 1/2;\ z^2/64]\ \ldots\ {}_0F_3[-;\ m_n+1/2,\ m_n+1,\ 1/2;\ z^2/64]\ldots(3.3)$$

Setting $\alpha_1 = \ldots = \alpha_n = 1/2 = \gamma_1 = \ldots = \gamma_n$ and $\beta_1 = \ldots = \beta_n = 1 = \delta_1 = \ldots = \delta_n$ in both sides of the theorem 2 and then making an appeal to the relation (1.22), we find

$$\left(\frac{4}{\pi}\right)^n \exp\left[-\frac{z^2}{4}\left\{\left(x_1^2+\frac{1}{x_1^2}\right)+\ldots+\left(x_n^2+\frac{1}{x_n^2}\right)\right\}\right]$$

$$\cdot \operatorname{Erfc}\left(-\frac{zx_1}{2}\right)\dots\operatorname{Erfc}\left(\frac{-zx_n}{2}\right)\cdot\operatorname{Erfc}\left(-\frac{z}{2x_1}\right)\dots\operatorname{Erfc}\left(-\frac{z}{2x_n}\right)$$

$$= \sum_{m_1,\dots,m_n=-\infty}^{\infty} x_1^{m_1}\dots x_n^{m_n}\left(\frac{z}{2}\right)^{m_1+\dots+m_n}\left(\sum_{t_1=0}^{\infty}\frac{\left(z^2/4\right)^{t_1}}{\frac{m_1+t_1}{2}!\,\frac{t_1}{2}!}\right)\dots\dots\left(\sum_{t_n=0}^{\infty}\frac{\left(z^2/4\right)^{t_n}}{\frac{m_n+t_n}{2}!\,\frac{t_n}{2}!}\right) \quad \dots\dots (3.4)$$

Setting $\alpha_1 = \dots\dots = \alpha_n = 2 = \gamma_1 = \dots\dots = \gamma_n$ and $\beta_1 = \dots\dots = \beta_n = 2 = \delta_1 = \dots\dots = \delta_n$ in both sides of the theorem 2 and then making an appeal to the relation (1.20), we derive

$$\left\{\sin h\sqrt{\frac{zx_1}{2}}\dots\dots\sin h\sqrt{\frac{zx_n}{2}}\right\}\left\{\sin h\sqrt{\frac{z}{2x_1}}\dots\dots\sin h\sqrt{\frac{z}{2x_n}}\right\}$$

$$= \sum_{m_1,\dots,m_n=-\infty}^{\infty} x_1^{m_1}\dots\dots x_n^{m_n}\frac{(z/4)^n(\pi)^{n/2}(z/8)^{m_1+\dots+m_n}}{\Gamma(m_1+1)\Gamma(m_1+3/2)\dots\dots\Gamma(m_n+1)\Gamma(m_n+3/2)}$$

$\times\ {}_0F_3\,[-;\ m_1+1,\ m_1+3/2,\ 3/2;\ z^2/64]\ \dots\ {}_0F_3\,[-;\ m_n+1\ m_n+3/2,\ 3/2;\ z^2/64]$ ...(3.5)

The result (3.4) may be rewritten as

$$\left(\frac{4}{\pi}\right)^n\cdot\exp\left[-\frac{z^2}{4}\left\{\sum_{i=1}^{n}\left(x_i+\frac{1}{x_i}\right)^2-2n\right\}\right]$$

$$\cdot \operatorname{Erfc}\left(-\frac{zx_1}{2}\right)\dots\dots\operatorname{Erfc}\left(\frac{-zx_n}{2}\right)\cdot\operatorname{Erfc}\left(-\frac{z}{2x_1}\right)\dots\dots\operatorname{Erfc}\left(-\frac{z}{2x_n}\right)$$

$$= \sum_{m_1,\dots,m_n=-\infty}^{\infty} x_1^{m_1}\dots\dots x_n^{m_n}\left(\frac{z}{2}\right)^{m_1+\dots+m_n}\left(\sum_{t_1=0}^{\infty}\frac{\left(z^2/4\right)^{t_1}}{\frac{m_1+t_1}{2}!\,\frac{t_1}{2}!}\right)\dots\dots\left(\sum_{t_n=0}^{\infty}\frac{\left(z^2/4\right)^{t_n}}{\frac{m_n+t_n}{2}!\,\frac{t_n}{2}!}\right) \dots\dots (3.6)$$

and

$$\left(\frac{4}{\pi}\right)^n\cdot\exp\left[-\frac{z^2}{4}\left\{\sum_{i=1}^{n}\left(x_i-\frac{1}{x_i}\right)^2+2n\right\}\right]\prod_{i=1}^{n}\operatorname{Erfc}\left(\frac{-z}{2x_i}\right)\prod_{i=1}^{n}\operatorname{Erfc}\left(\frac{-zx_i}{2}\right)$$

$$= \sum_{m_1,\dots,m_n=-\infty}^{\infty} x_1^{m_1}\dots\dots x_n^{m_n}\left(\frac{z}{2}\right)^{m_1+\dots+m_n}\left(\sum_{t_1=0}^{\infty}\frac{\left(z^2/4\right)^{t_1}}{\frac{m_1+t_1}{2}!\,\frac{t_1}{2}!}\right)\dots\dots\left(\sum_{t_n=0}^{\infty}\frac{\left(z^2/4\right)^{t_n}}{\frac{m_n+t_n}{2}!\,\frac{t_n}{2}!}\right) \quad \dots (3.7)$$

Setting $\alpha_1 = \ldots\ldots = \alpha_n = 2 = \gamma_1 = \ldots\ldots = \gamma_n$ and $\beta_1 = \ldots = \beta_n = 1 = \delta_1 = \ldots = \delta_n$ in both sides of the theorem 3 and then making an use of the relations (1.17) and (1.19), we obtain

$$\left\{\cos h\sqrt{\frac{zx_1}{n+1}}\ldots\ldots\cos h\sqrt{\frac{zx_n}{n+1}}\right\}\left\{\cos\sqrt{\frac{z}{(n+1)(x_1\ldots\ldots x_n)}}\right\}^n$$

$$= \sum_{m_1\ldots\ldots m_n=-\infty}^{\infty} x_1^{m_1}\ldots\ldots x_n^{m_n}\frac{(\pi)^{n/2}\left(\frac{z}{4(n+1)}\right)^{m_1+\ldots+m_n}}{\Gamma(m_1+1)\Gamma(m_1+1/2)\ldots\ldots\Gamma(m_n+1)\Gamma(m_n+1/2)}$$

$$\times F^{0:0;\ldots\ldots;0}_{2n:1;\ldots\ldots;1}\left(\begin{matrix}(-) : (-);\ldots\ldots;(-);\\ \left(m_n+1, m_n+\frac{1}{2}\right):\left(\frac{1}{2}\right);\ldots\ldots;\left(\frac{1}{2}\right);\end{matrix} -\left(\frac{z}{4(n+1)}\right)^{n+1},\ldots\ldots,-\left(\frac{z}{4(n+1)}\right)^{n+1}\right)$$

provided that $\left|\left(\frac{z}{4(n+1)}\right)^{n+1}\right| < 1$ and all $x_i \neq 0$, $i = 1, \ldots, n$. ..... (3.8)

where $F^{p:q_1;\ldots\ldots;q_n}_{l:m_1;\ldots\ldots;m_n}(z_1, \ldots z_n)$ is the multivariable generalized Kample' de Fe'riet function due to Srivastava and Panda [12, p. 1127, eq (4.1) et sq.]. (Also, see Srivastava and Manocha [11, p.65, eqs (21–25)]).

Setting $\alpha_1 = \ldots\ldots = \alpha_n = 1/2$, $\beta_1 = \ldots\ldots = \beta_n = 1$ and $\gamma_1 = \ldots\ldots = \gamma_n = 2$, and $\delta_1 = \ldots = \delta_n = 1$ in both sides of the theorem 3 and then making an appeal to the relations (1.17) and (1.22), we obtain

$$\left(\frac{2}{\sqrt{\pi}}\cos\sqrt{\frac{z}{(n+1)(x_1\ldots\ldots x_n)}}\right)^n\cdot\exp\left[\frac{-z^2}{(n+1)^2}\left(x_1^2+\ldots+x_n^2\right)\right]$$

$$\cdot\,\mathrm{Erfc}\left(\frac{-zx_1}{n+1}\right)\ldots\ldots\mathrm{Erfc}\left(\frac{-zx_n}{n+1}\right) = \sum_{m_1\ldots,m_n=-\infty}^{\infty} x_1^{m_1}\ldots\ldots x_n^{m_n}\left(\frac{z}{n+1}\right)^{m_1+\ldots+m_n}$$

$$\times S^{0:1;\ldots;1}_{n:1;\ldots;1}\left[\begin{matrix}[-----]:[1:1];\ldots\ldots;[1:1];\\ \left[\frac{m_1}{2}+1:\frac{1}{2};\ldots;\frac{1}{2}\right]\ldots\ldots\left[\frac{m_n}{2}+1:\frac{1}{2};\ldots\frac{1}{2}\right]:[1:2];\ldots;[1:2];\end{matrix} -\left(\frac{z}{n+1}\right)^{n+1},\ldots-\left(\frac{z}{n+1}\right)^{n+1}\right]$$

..... (3.9)

where, the multiple hypergeometric series $S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ is known as generalized Lauricella series in several variables and is defined by Srivastava and Daoust [9, 10. p. 454].

Setting $\alpha_1 = \dots = \alpha_n = 1/2 = \gamma_1 = \dots = \gamma_n$ and $\beta_1 = \dots = \beta_n = 1$, $\delta_1 = \dots = \delta_n = 1$ in both sides of the theorem 4 and then making an appeal to the relation (1.22) we derive

$$\left(\frac{4}{\pi}\right)^n \cdot \exp\left[\frac{-z^2}{(n+1)^2}\left(x_1^2 + \dots + x_n^2 + \frac{n}{x_1^2 \dots x_n^2}\right)\right]$$

$$\cdot \operatorname{Erfc}\left(\frac{-zx_1}{n+1}\right) \dots \operatorname{Erfc}\left(\frac{-zx_n}{n+1}\right)\left\{\operatorname{Erfc}\left(\frac{-z}{(n+1)(x_1 \dots x_n)}\right)\right\}^n$$

$$= \sum_{m_1, \dots, m_n = -\infty}^{\infty} x_1^{m_1} \dots x_n^{m_n}\left(\frac{z}{n+1}\right)^{m_1 + \dots + m_n} \times$$

$$\times S^{0:1;\dots;1}_{n:1;\dots;1}\left[\begin{matrix} [-----] \dots : [1:1]; \dots, [1:1]; \\ \left[\frac{m_1}{2}+1:\frac{1}{2};\dots;\frac{1}{2}\right], \dots, \left[\frac{m_n}{2}+1:\frac{1}{2};\dots\frac{1}{2}\right]:\left[1:\frac{1}{2}\right];\dots;\left[1:\frac{1}{2}\right]; \end{matrix} \left(\frac{z}{n+1}\right)^{n+1}, \dots, \left(\frac{z}{n+1}\right)^{n+1}\right] \quad \dots\dots(3.10)$$

Similarly setting various other values of α_i, β_i, γ_i and δ_i, $i = 1, \dots, n$ in both sides of the theorems 1 – 4 and then making an appeal to the result (1.16) to (1.22), we may evaluate different more generating functions.

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Jour. Pure Math., Vol. 23, 2006, pp. 81-88

A NOTE ON CONTRA-δ-PRECONTINUITY

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ABSTRACT : In this paper, a kind of continuity termed contra-δ-precontinuity is considered via the concept of δ-preopen set. The relationship of this type of function with a few other existing allied types of functions are also studied.

Key words and pharases : kernel, δ-preopen, δ-preclosure, contra-δ-precontinuous function, contra-δ-pregraph.

2005 AMS Subject Classification Code : 54C05, 54D10

1. INTRODUCTION

In 1968, Velicko [12] introduced δ-open sets and in 1993, Roychowdhury and Mukherjee [9] introduced the notions of δ-prepen sets and δ-preclosures. Since then δ-preopen sets have been widely used in order to introduce new types of spaces and functions.

In this paper, we define a type of continuity termed contra-δ-precontinuity by using the notions of δ-preopen sets and δ-preclosure operator. Some basic properties of this new class of functions are investigated. The relationships of this type of functions with a few other existing allied types of functions are also studied. Finally, certain relationships among contra-δ-precontinuity, connectedness, graph and some separation axioms are investigated.

Henceforth by spaces X, Y (or simply by X, Y) we mean topological spaces. We shall write $f : X \to Y$ to denote a function f from a space X to a space Y. For a subset A of a space X, cl$A$ and int$A$ denote as usual, the closure and interior of A respectively. A subset A of a space X is said to be regular open (resp. regular closed) if $A = \text{intcl}A$ (resp. $A = \text{clint}A$). A set A of (X, τ) is called δ-closed [12] if $A = \text{cl}_\delta A$, where $\text{cl}_\delta A = \{x \in X : \text{intcl}U \cap A \neq \phi$ for all $U \in \tau$ with $x \in U\}$. The complement of a δ-closed set is known as a δ-open set. A subset A of a space (X, τ) is called δ-preopen [9] if $A \subseteq \text{intcl}_\delta A$. The complement of δ-preopen sets is said to be δ-preclosed. For any $A \subseteq X$, the union (intersection) of all δ-preopen (δ-preclosed) sets in X, each contained in (containing) A is called the δ-preinterior (resp. δ-preclosure) of A in X denoted by $\text{pint}_\delta A$ (resp. $\text{pcl}_\delta A$) [9]. It is also well known that a subset A of a space X is δ-preopen iff for each $x \in A$, there exists a δ-preopen set U such that $x \in U \subseteq A$ [9]. Again it is known that $x \in \text{pcl}_\delta A$ iff for each δ-preopen set U containing x, $U \cap A \neq \phi$.

The family of all δ-preopen (resp. δ-preclosed, δ-open, regular open) sets of a space X is denoted by $\delta\text{-}PO(X)$ (resp. $\delta\text{-}PC(X)$, $\delta\text{-}O(X)$, $RO(X)$) while $\delta\text{-}PO(x)$ is the collection of all δ-preopen sets of X containing a point $x \in X$. The above stated definitions and results (and further relevant details) are well known by now, and can be had, for instance, in [2].

2. CONTRA-δ-PRECONTINUITY AND ITS APPLICATIONS

Definition 2.1. [4] Let A be a subset of a topological space (X, τ). Then the kernel of A is denoted by $ker(A)$ and is defined by $\cap\{U \in \tau : A \subseteq U\}$.

Observation 2.2. Let (X, τ) be a topological space and $A \subseteq X$. Then one can check in a straightforward manner that

(a) $x \in ker(A)$ iff $A \cap F \neq \phi$ for any closed set F containing x.

(b) $A \subseteq B \Rightarrow ker(A) \subseteq ker(B)$.

(c) $A \subseteq ker(A)$ and if A is open, then $A = ker(A)$.

Definition 2.3. A function $f : X \to Y$ is said to be conta-δ-precontinuous at a point $x \in X$ if for each closed set F in Y with $f(x) \in F$, there exists $U \in \delta\text{-}PO(x)$ such that $f(U) \subseteq F$. f is called contra-δ-precontinuous (on X) if it is so at each point of X.

Theorem 2.4. For a function $f : X \to Y$ the following are equivalent:

(i) f is conta-δ-precontinuous on X.

(ii) $f^{-1}(F) \in \delta\text{-}PO(X)$, for every closed set F in Y.

(iii) $f^{-1}(U) \in \delta\text{-}PC(X)$, for every open set U in Y.

(iv) $f(\text{pcl}_\delta A) \subseteq ker(f(A))$, for any $A \subseteq X$.

(v) $\text{pcl}_\delta(f^{-1}(B)) \subseteq f^{-1}(ker(B))$, for any $B \subseteq Y$.

Proof. (i) $\Rightarrow$ (ii) : Let F be a closed subset of Y with $x \in f^{-1}(F)$, so that $f(x) \in F$. Then by contra-δ-continuity of f, there exists $U \in \delta\text{-}PO(x)$ such that $f(U) \subseteq F$. Then $x \in U \subseteq f^{-1}(F)$. So $f^{-1}(F) \in \delta\text{-}PO(X)$.

(ii) $\Rightarrow$ (iii) : Let U be any open set in Y. Then $Y \setminus U$ is a closed set in Y. Hence by (ii), $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U) \in \delta\text{-}PO(X)$ i.e., $f^{-1}(U) \in \delta\text{-}PC(X)$.

(iii) $\Rightarrow$ (i) : Let $x \in X$ and F be any closed set in Y with $f(x) \in F$. Then by (iii), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \in \delta\text{-}PC(X) \Rightarrow f^{-1}(F) \in \delta\text{-}PO(X)$. Let $U = f^{-1}(F)$. Then $x \in U$ and $f(U) \subseteq F$.

(ii) $\Rightarrow$ (iv) : Let $A \subseteq X$ and $y \in f(\text{pcl}_\delta A)$, but $y \notin ker(f(A))$. Then by Observation 2.2(i), there exists a closed set F containing y such that $f(A) \cap F = \phi$. Hence $A \cap f^{-1}(F) = \phi$. Now by (ii), $f^{-1}(F) \in \delta\text{-}PO(X)$, where $x \in f^{-1}(y)$. Thus $x \notin \mathrm{pcl}_\delta A$ and hence $f(x) = y \notin f(\mathrm{pcl}_\delta A)$, a contradiction. Thus $f(\mathrm{pcl}_\delta A) \subseteq ker(f(A))$.

(iv) ⇒ (v) : Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$. Then by (iv), $f(\mathrm{pcl}_\delta(f^{-1}(B))) \subseteq ker(f(f^{-1}(B))) \subseteq ker(B) \Rightarrow \mathrm{pcl}_\delta f^{-1}(B) \subseteq f^{-1}(ker(B))$.

(v) ⇒ (iii) : Let U be any open subset of Y. Then by (v), we have, $\mathrm{pcl}_\delta(f^{-1}(U) \subseteq f^{-1}(ker(U))) = f^{-1}(U)$ (by Observation 2.2(iii)). Thus $f^{-1}(U) = \mathrm{pcl}_\delta(f^{-1}(U))$ i.e., $f^{-1}(U) \in \delta\text{-}PO(X)$.

Definition 2.5. A function $f : X \to Y$ is said to be

(i) δ-continuous [6] if $f^{-1}(V)$ is δ-open in X for every $V \in RO(Y)$;

(ii) δ*-almost continuous [8] if $f^{-1}(U) \in \delta\text{-}PO(X)$, $\forall\ U \in \delta\text{-}PO(Y)$; or equivalently, for each $x \in X$ and each $V \in \delta\text{-}PO(f(x))$, there exists a $W \in \delta\text{-}PO(x)$ such that $f(W) \subseteq V$;

(iii) supercontinuous [5, 10] if $f^{-1}(V)$ is δ-open in X for every open set V in Y;

(iv) RC-continuous [1] if $f^{-1}(V)$ is regular closed in X for each open set V in Y.

Remark 2.6. From the definition of RC-continuity and contra-δ-precontinuity it follows that RC-continuity ⇒ contra-δ-precontinuity. However, we show in the following example that a contra-δ-precontinuous function need not be any of the above types of functions.

Example 2.7. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then (X, τ) is a topological space such that $RO(X) = \tau = \delta\text{-}O(X)$ and $\delta\text{-}PO(X) = \mathcal{P}(X)$. Define a map $f : (X, \tau) \to (X, \tau)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. $\{a\}$ is open in (X, τ) but $f^{-1}(\{a\}) = \{c\} \notin RC(X)$. Thus f is not RC-continuous. $\{b, c\} \in RO(X)$ but $f^{-1}(\{b, c\}) = \{a, b\} \notin \delta\text{-}O(X)$. Thus f is not δ-continuous and f is not super continuous (as $f^{-1}(\{b, c\}) \notin \delta\text{-}O(X)$ as $\{b, c\} \in \tau$). But f is clearly contra-δ-precontinuous.

Example 2.8. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}. \{a, c\}, X\}$. The (X, τ) and (Y, σ) are two topological spaces such that the collection of δ-preopen subsets of (X, τ) is τ and that of all δ-preopen subsets of (X, σ) is $\mathcal{P}(X)$. Consider the map $f : (X, \tau) \to (X, \sigma)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Clearly f is contra-δ-precontinuous. But f is not δ*-almost continuous as $\{a, b\}$ is δ-preopen in (X, σ) but $f^{-1}(\{a, b\}) = \{a, c\}$ is not δ-preopen in (X, τ). But f is not δ*-almost continuous as $\{a, b\}$ is δ-preopen in $(X. \sigma)$ but $f^{-1}(\{a, b\}) = \{a, c\}$ is not δ-preopen in (X, τ).

Example 2.9. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then (X, τ) is a topological space such that $\tau = \delta\text{-}PO(X)$. Consider the identity map $f : (X, \tau) \to (X, \tau)$. Then f is δ*-almost continuous map but $\{b, c\}$ is a closed set in (X, τ) and $f^{-1}(\{b, c\}) = \{b, c\} \notin \delta\text{-}PO(X)$ and hence f is not contra-δ-precontinuous.

Next example shows δ-continuous (super continuous) function may not be contra-δ-precontinuous.

Example 2.10. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\tau = \delta\text{-}O(X)$ and $RO(X) = \{\phi, X\}$. Consider the identity map $f : (X, \tau) \to (X, \tau)$. Then f is δ-continuous as well as super continuous. But it is easy to see that f is not contra-δ-precontinuous.

Theorem 2.11. Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

(i) f is δ*-almost continuous and g is contra-δ-precontinuous $\Rightarrow g_o f : X \to Z$ is contra-δ-precontinuous;

(ii) f is contra-δ-precontinuous and g is continuous $\Rightarrow g_o f : X \to Z$ is contra-δ-precontinuous;

(iii) f is super continuous and g is RC-continuous $\Rightarrow g_o f : X \to Z$ is contra-δ-precontinuous;

(iv) f is δ-continuous and g is RC-continuous $\Rightarrow g_o f . X \to Z$ is contra-δ-precontinuous.

Proof. The proof being straightforward, is omitted.

Definition 2.12. A function $f : X \to Y$ is called δ-preopen if image of each δ-preopen set is δ-preopen.

Theorem 2.13. Let $f : X \to Y$ be a surjective δ-preopen function and $g : Y \to Z$ be a function such that $g_o f : X \to Z$ is contra-δ-precontinuous, then g is contra-δ-precontinuous.

Proof. Let $y \in Y$. Then by the ontoness of f, there exists $x \in X$ such that $y = f(x)$. Let F be a closed set in Z containing $g(f(x))$. Then there exists a δ-preopen set U in X containing x such that $g(f(U)) \subseteq F$. Since f is δ-preopen, $f(U)$ is δ-preopen in Y containing y such that $g(f(U)) \subseteq F$. Thus g is contra-δ-precontinuous.

Corollary 2.14. Let $f : X \to Y$ be a δ*-almost continuous, δ-preopen surjection and $f : Y \to Z$ be any function. Then $g_o f : X \to Z$ is contra-δ-precontinuous iff g is contra-δ-precontinuous.

Proof. follows from Theorems 2.11(i) and 2.13.

Lemma 2.15. [9] (i) Let A and X_0 be two subsets of a space (X, τ). If $A \in \delta\text{-}PO(X)$ and $X_0 \in \delta\text{-}O(X)$, then $A \cap X_0 \in \delta\text{-}PO(X_0)$.

(ii) $A \subseteq X_0 \subseteq X$. If $X_0 \in \delta\text{-}O(X)$ and $A \in \delta\text{-}PO(X_0)$, then $A \in \delta\text{-}PO(X)$.

Theorem 2.16. If $f : X \to Y$ is a contra-δ-precontinuous function and A is any δ-open subset of X, then $f|_A : A \to Y$ is contra-δ-precontinuous.

Proof. Let F be a closed set in Y. Then $f^{-1}(F) \in \delta\text{-}PO(X)$ (by Theorem 2.4(ii)). Now $A \in \delta\text{-}O(X)$, so $(f|_A)^{-1}(F) = A \cap f^{-1}(F) \in \delta\text{-}PO(A)$ (by Lemma 2.15).

Theorem 2.17. Let $f : X \to Y$ be a function and $\{U_\alpha : \alpha \in \Lambda\}$ be a δ-open cover of X. If for each $\alpha \in \Lambda$ $f|_{U_\alpha}$ is contra-δ-precontinuous, then so is $f : X \to Y$.

Proof. Let F be a closed set in Y. Since $f|_{U_\alpha}$ is contra-δ-precontinuous for each $\alpha \in \Lambda$, $(f|_{U_\alpha})^{-1}(F) \in \delta\text{-}PO(U_\alpha)$ (by Theorem 2.4(ii). Since $U_\alpha \in \delta\text{-}O(X)$, by Lemma 2.15, $(f|_{U_\alpha})^{-1}(F) \in \delta\text{-}PO(X)$ for each $\alpha \in \Lambda$. Now $f^{-1}(F) = \cup_{\alpha\in\Lambda}(f|_{U_\alpha})^{-1}(F) \in \delta\text{-}PO(X)$. Thus f is contra-δ-precontinuous.

Definition 2.18. A space X is called

(i) δ-preconnected if X cannot be expressed as a union of two disjoint non-empty δ-preopen sets;

(ii) δ-pre ultraconnected if every two non-viod δ-preclosed sets of X intersect;

(iii) hyperconnected [7] if every open set is dense in X.

Remark 2.19. Every δ-pre ultraconnected space is δ-preconnected.

The next two examples show that that hyperconnectedness is independent of the other two concepts of connectedness defined above.

Example 2.21. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\tau = \delta\text{-}PO(X)$. So X is δ-pre ultraconnected and hence by Remark 2.19, it is δ-preconnected but X is not hyperconnected.

Example 2.22. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ) is hyperconnected but not δ-preconnected (as $\delta\text{-}PO(X) = \mathcal{P}(X)$) and hence not δ-pre ultraconnected.

Theorem 2.23. Let $f : X \to Y$ be a contra-δ-precontinuous surjection and X be δ-preconnected, then Y is connected.

Proof. Straightforward.

Theorem 2.24. If X is δ-pre ultraconnected and $f : X \to Y$ is a contra-δ-precontinuous surjection, then Y is hyperconnected.

Proof. Suppose Y is not hyperconnected. Then there exists an open set V such that V is not dense in Y. Let $B_1 = \text{intcl}V$, $B_2 = Y \setminus \text{cl}V$. Then B_1 and B_2 are nonempty disjoint open sets in Y and hence $f^{-1}(B_1)$, $f^{-1}(B_2)$ are disjoint nonempty δ-preclosed subsets of X - a contradiction to the δ-pre ultraconnectedness of X. Hence Y is hyperconnected.

3. CONTRA-δ-PREGRAPH AND SEPARATION AXIOMS

Definition 3.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be a contra-δ-pregraph if for each $(x, y) \in X \times Y \setminus G(f)$, there exist some $U \in \delta\text{-}PO(x)$ and a closed set V in Y containing y such that $(U \times V) \cap G(f) = \phi$.

Proposition 3.2. The following statements are equivalent for the graph $G(f)$ of a function $f : X \rightarrow Y$:

(i) $G(f)$ is a contra-δ-pregraph.

(ii) For each $(x, y) \in X \times Y \setminus G(f)$, there exists $U \in \delta\text{-}PO(x)$ and closed set V in Y containing y such that $f(U) \cap V = \phi$.

Proof. Obvious.

Theorem 3.3. If $f : X \rightarrow Y$ is contra-δ-precontinuous and Y is Urysohn then $G(f)$ is a contra-δ-pregraph in $X \times Y$.

Proof. Suppose Y is Urysohn and $(x, y)\ X \times Y \setminus G(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $y \in W$ and $\text{cl}V \cap \text{cl}W = \phi$. Since f is contra-δ-precontinuous, there exists a δ-preopen set U in X containing x such that $f(U) \subseteq \text{cl}V$. Thus $f(U) \cap \text{cl}W = \phi$ and hence $G(f)$ is a contra-δ-pregraph in $X \times Y$.

Theorem 3.4. Let $f: X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ be the graph of f, defined by $g(x) = (x, f(x))$ for each $x \in X$. If g is contra-δ-precontinuous, then so is f.

Proof. Let U be an open set in Y, then $X \times U$ is open in $X \times Y$. It then follows that $f^{-1}(U) = g^{-1}(X \times U) \in \delta\text{-}PC(X)$ (as g is contra-δ-precontinuous). Hence f is contra-δ-precontinuous.

Definition 3.5. [3] A function $f : X \rightarrow Y$ is contra super-continuous if for each $x \in X$ and each closed set F containing $f(x)$, there exists a regular open set U in X containing x such that $f(U) \subseteq F$.

Theorem 3.6. If $f: X \rightarrow Y$ is contra super-continuous, $g : X \rightarrow Y$ is contra-δ-precontinuous and Y is Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is δ-preclosed in X.

Proof. If $x \in X \setminus E$ then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets V and W such $f(x) \in V$, $g(x) \in W$ and $\text{cl}U \cap \text{cl}W = \phi$. Since f is contra-super-continuous and g is contra-δ-precontinuous, there exists $U \in RO(X)$ containing x and $G \in \delta\text{-}PO(x)$ such $f(U) \subseteq \text{cl}V$ and $g(G) \subseteq \text{cl}W$. Let $O = U \cap G$. Then O is δ-preopen in X (by Lemma 2.15) containing x and $f(O) \cap g(O) = \phi$. Then $O \cap E = \phi$ and hence $x \notin \text{pcl}_\delta E$. Thus E is δ-preclosed in X.

Definition 3.7. [11] A space X is said to be weakly Hausdorff if each element of X is an intersection of some regular closed sets.

Definition 3.8. [2] A space X is said to be

(i) (δ, p)-T_1 if for any pair of distinct points $x, y \in X$, there exist a δ-preopen set U in X containing x but not containing y and a δ-preopen set V in X containing y but not x.

(ii) (δ, p)-T_2 if for any pair of distinct points $x, y \in X$, there exist δ-preopen sets U and V in X containing x and y respectively such that $U \cap V = \phi$.

Theorem 3.9. If $f: X \to Y$ is contra-δ-precontinuous, injection and Y is weakly Hausdorff, then X is (δ, p)-T_1.

Proof. Suppose Y is weakly Hausdorff. For any distinct points $x, y \in X$, $f(x) \neq f(y)$ and so there exists regular closed sets V and W in Y such that $f(x) \in V$, $f(y) \notin V$, $f(y) \in W$ and $f(x) \notin W$. Since f is contra-δ-precontinuous, $f^{-1}(V)$, $f^{-1}(W) \in \delta\text{-}PO(X)$ such that $x \in f^{-1}(V)$ and $y \in f^{-1}(W)$ and $x \notin f^{-1}(W)$. Thus X is (δ, p)-T_1.

Theorem 3.10. Let $f: X \to Y$ is contra-δ-precontinuous injection function having contra-δ-pregraph. Then X is (δ, p)-T_1.

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$ (as f is injective). Thus $(x, f(y)) \in X \times Y \setminus G(f)$. Then by Proposition 3.2, there exists $U \in \delta\text{-}PO(x)$ and a closed set F in Y containing $f(y)$ such that $f(U) \cap F = \phi$. Hence $U \cap f^{-1}(F) = \phi$. Since f is contra-δ-precontinuous $f^{-1}(F) \in \delta\text{-}PO(x)$ (by Theorem 2.4). Also $y \in f^{-1}(F)$ but $x \notin f^{-1}(F)$ and $x \in U$ and $y \notin U$. Thus X is (δ, p)-T_1.

Theorem 3.11. Let $f: X \to Y$ is contra-δ-precontinuous injection and Y be Urysohn. Then X is (δ, p)-T_2.

Proof. Suppose Y is Urysohn and x, y be two distinct points of X. Then $f(x) \neq f(y)$ (as f is injective). Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $f(y) \in W$ and $\mathrm{cl}V \cap \mathrm{cl}W = \phi$. Since f is contra-δ-precontinuous, there exist δ-preopen sets U and G in X containing x and y respectively such that $f(U) \subseteq \mathrm{cl}V$ and $f(G) \subseteq \mathrm{cl}W$. Hence $G \cap U = \phi$. Thus X is (δ, p)-T_2.

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Jour. Pure Math., Vol. 23, 2006, pp. 89-94

RELATIVE ASYMPTOTIC REGULARITY AND COMMON FIXED POINTS IN 2-METRIC SPACE

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1. INTRODUCTION

The notion of compatibility is given by Jungck (1986); Rhoades et al. (1984) used asymptotic regularity for a pair of maps.

In this paper we prove a common fixed point theorem using the concept of asymptotic regularity of sequence in 2-metric space.

In the proof of our theorem we need the following definitions.

Definition 1. Let A and S be two self maps of a 2-metric space (X, d) and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be asymptotically S-regular with respect to A if

$$\lim_{n \to \infty} d(Ax_n, Sx_n, a) = 0$$

Definition 2. The pair $\{A, S\}$ of self maps of 2-metric space (X, d) is said to be compatible if

$$\lim_{n \to \infty} d(ASx_n, SAx_n, a) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \text{ for some } t \in X$$

Definition 3. An algebraic structure (X, d) is called a 2 – metric space where X is a non-empty set and d is a positive real valued function defined on $X \times X \times X$ satisfying

1. $d(x, y, z) \neq 0$
2. $d(x, y, z) = 0$ if atleast two are equal
3. $d(x, y, z) = d(x, z, y) = d(y, z, x)$
4. $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z) \ \forall\ a \in X$

This concept was given by Gahler and geometrically it gives the area of triangle.

Definition 4. A sequence $\{x_n\}$ is said to be a Cauchy Sequence in a 2-metric space X if

$$\lim_{m,\, n \to \infty} d(x_m, x_n, a) = 0.$$

Definition 5. Any 2-metric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to some point $z \in X$.

MAIN RESULT

Let A, B, S and T be self maps of complete 2-metric space (X, d) satisfying the following conditions.

i. $$d(Ax, By, a) \leq \frac{a_1 d(Ty, By, a)[1 + d(Sx, Ax, a)]}{[1 + d(Sy, Ay, a)]}$$
$$+ a_2[d(Sx, Ax, a) + d(Ty, By, a)]$$
$$+ a_3[d(Sx, By, a) + d(Ty, Ax, a)]$$
$$+ a_4 d(Sx, Ty, a)$$

where

$$0 \leq \max\{a_1 + a_3 + a_4,\ a_2 + 2a_3 + a_4,\ a_1 + a_2 + a_3\} < 1$$

and $a_i \geq 0$, $i = 1, 2, 3, 4$

ii. $\{A, S\}$ is a compatible pair of maps

iii. S is continuous

iv. $d(x, Tx, a) \leq d(x, Sx, a)$ for all $x \in X$

v. $\{x_n\}$ is asymptotically A-regular with respect to S and $\{y_n\}$ is asymptotically B-regular with respect to T.

Then A, B, S and T have a unique common fixed point.

Proof. For any positive integers m, n we have

$$d(Ax_m, By_n, a) \leq \frac{a_1 d(Ty_n, By_n, a)[1 + d(Sx_m, Ax_m, a)]}{[1 + d(Sx_m, Ty_n, a)]}$$
$$+ a_2[d(Sx_m, Ax_m, a) + d(Ty_n, By_n, a)]$$
$$+ a_3[d(Sx_m, By_n, a) + d(Ty_n, Ax_m, a)]$$
$$+ a_4 d(Sx_m, Ty_n, a)$$

$$
\begin{aligned}
&\le \frac{a_1 d(Ty_n, By_n, a)\left[1 + d(Sx_m, Ax_m, a)\right]}{\left[1 + d(Sx_m, Ty_n, a)\right]}\\
&\qquad + a_2 d(Sx_m, Ax_m, a) + a_2 d(Ty_n, By_n, a)\\
&\qquad + a_3 d(Sx_m, Ax_m, a) + a_3 d(Ax_m, By_n, a)\\
&\qquad + a_3 d(Sx_m, By_n, Ax_m) + a_3 d(By_n, Ax_m, a)\\
&\qquad + a_3 d(Ty_n, By_n, a) + a_3 d(Ty_n, Ax_m, By_n)\\
&\qquad + a_4 d(Sx_m, Ax_m, a) + a_4 d(Ax_m, Ty_n, a)\\
&\qquad + a_4 d(Sx_m, Ty_m, Ax_m)\\
&\le \frac{a_1 d(Ty_n, By_n, a)\left[1 + d(Sx_m, Ax_m, a)\right]}{\left[1 + d(Sx_m, Ty_n, a)\right]}\\
&\qquad + a_2 d(Sx_m, Ax_m, a) + a_2 d(Ty_n, By_n, a)\\
&\qquad + a_3 d(Sx_m, Ax_m, a) + a_3 d(Ax_m, By_n, a)\\
&\qquad + a_3 d(Sx_m, By_n, Ax_m) + a_3 d(By_n, Ax_m, a)\\
&\qquad + a_3 d(Ty_n, By_n, a) + a_3 d(Ty_n, Ax_m, By_n)\\
&\qquad + a_4 d(Sx_m, Ax_m, a) + a_4 d(Sx_m, Ty_n, Ax_m)\\
&\qquad + a_4 d(Ax_m, By_n, a) + a_4 d(By_n, Ty_n, a)\\
&\qquad + a_4 d(Ax_m, Ty_m, By_m)
\end{aligned}
$$

On further simplification, we get

$$
\begin{aligned}
d(Ax_m, By_n, a) \le{}& \frac{a_1 d(Ty_n, By_n, a)\left[1 + d(Sx_m, Ax_m, a)\right]}{(1 - 4a_3 - 2a_4)\left[1 + d(Sx_m, Ty_n, a)\right]}\\
&+ \frac{(a_2 + a_3 + 2a_4)}{(1 - 4a_3 - 2a_4)} d(Sx_m, Ax_m, a)\\
&+ \frac{(a_2 + a_3 + a_4)}{(1 - 4a_3 - 2a_4)} d(Ty_n, By_n, a)
\end{aligned}
$$

Now

$$
\begin{aligned}
d(Ax_m, Ax_n, a) &\le d(Ax_m, By_n, a) + d(By_n, Ax_n, a) + d(Ax_m, Ax_n, By_n)\\
&\le d(Ax_m, By_n, a) + 2d(Ax_n, By_n, a)
\end{aligned}
$$

Hence we have

$$d(Ax_m, Ax_n, a) \le \frac{a_1 d(Ty_n, By_n, a)\left[1 + d(Sx_m, Ax_m, a)\right]}{(1 - 4a_3 - 2a_4)\left[1 + d(Sx_m, Ty_n, a)\right]}$$

$$+\frac{(a_2 + a_3 + 2a_4)}{(1 - 4a_3 - 2a_4)} d(Sx_m, Ax_m, a)$$

$$+\frac{(a_2 + a_3 + a_4)}{(1 - 4a_3 - 2a_4)} d(Ty_n, By_n, a)$$

$$+\frac{2a_1 d(Ty_n, By_n, a)\left[1 + d(Sx_n, Ax_n, a)\right]}{(1 - 4a_3 - 2a_4)\left[1 + d(Sx_n, Ty_n, a)\right]}$$

$$+\frac{2(a_2 + a_3 + a_4) d(Sx_n, Ax_n, a)}{(1 - 4a_3 - 2a_4)}$$

$$+\frac{2(a_2 + a_3 + a_4) d(Ty_n, By_n, a)}{(1 - 4a_3 - 2a_4)}$$

Taking limit as $n, m \to \infty$ and using condition (5), we see that $\{Ax_n\}$ is a Cauchy sequence. Hence $\{Ax_n\}$ converges to some point z in X.

Now

$d(Sx_n, z, a) \le d(Ax_n, z, a) + d(Sx_n, Ax_n, a) + d(Sx_n, z, Ax_n)$.

Taking limit as $n \to \infty$, we see that $\{Sx_n\}$ also converges to z.

In the same way we can prove $\{By_n\}$ and $\{Ty_n\}$ also converge to z.

By condition (3), we see that $\{S^2x_n\}$ and $\{SAx_n\}$ converge to Sz.

Now

$d(ASx_n, Sz, a) \le d(ASx_n, SAx_n, a) + d(SAx_n, Sz, a) + d(ASx_n, Sz, SAx_n)$.

Since $\{A, S\}$ is compatible pair of maps so $\{ASx_n\} \to Sz$.

Further by condition (1), we have

$$d(ASx_n, By_n, a) \le \frac{a_1 d(Ty_n, By_n, a)\left[1 + d(S^2x_n, ASx_n, a)\right]}{\left[1 + d(S^2x_n, Ty_n, a)\right]}$$

$$+ a_2[d(S^2x_n, ASx_n, a) + d(Ty_n, By_n, a)]$$
$$+ a_3[d(S^2x_n, By_n, a) + d(Ty_n, ASx_n, a)]$$
$$+ a_4d(S^2x_n, Ty_n, a).$$

Taking limit as $n \to \infty$, we obtain

$$d(Sz, z, a) \le (2a_3 + a_4)d(Sz, z, a)$$

which is a contradiction. Hence

$$Sz = z.$$

By condition (4), we have

$$d(z, Tz, a) \le d(z, Sz, a)$$
$$= 0.$$

Hence $Tz = z$.

Again using condition (1), we have

$$d(Az, By_n, a) \le \frac{a_1 d(By_n, Ty_n, a)[1 + d(Sz, Az, a)]}{[1 + d(Sz, Ty_n, a)]}$$
$$+ a_2[d(Sz, Az, a) + d(Ty_n, By_n, a)]$$
$$+ a_3[d(Sz, By_n, a) + d(Ty_n, Az, a)]$$
$$+ a_4d(Sz, Ty_n, a).$$

Taking limit as $n \to \infty$, we have

$d(Az, z, a) \le (a_2 + a_3)d(Az, z, a)$ which is a contradiction.

Hence $Az = z$.

Further

$$d(z, Bz, a) = d(Az, Bz, a)$$
$$\le \frac{a_1 d(Tz, Bz, a)[1 + d(Sz, Az, a)]}{[1 + d(Sz, Tz, a)]}$$
$$+ a_2[d(Sz, Az, a) + d(Tz, Bz, a)]$$
$$+ a_3[d(Sz, Bz, a) + d(Tz, Az, a)]$$
$$+ a_4d(Sz, Tz, a)$$

which implies

$$d(z, Bz, a) \leq (a_1 + a_2 + a_3)d(z, Bz, a)$$

$$i.e.\ Bz = z$$

Thus z is the common fixed point of A, B, S and T.

Next, we will prove the uniqueness of z; let w be another common fixed point, then we have

$$\begin{aligned} d(w, z, a) &= d(Aw, Bz, a) \\ &\leq \frac{a_1 d(Tz, Bz, a)\left[1 + d(Sw, Aw, a)\right]}{\left[1 + d(Sw, Tz, a)\right]} \\ &\quad + a_2[d(Sw, Aw, a) + d(Tz, Bz, a)] \\ &\quad + a_3[d(Sw, Bz, a) + d(Tz, Aw, a)] \\ &\quad + a_4 d(Sw, Tz, a) \end{aligned}$$

which implies

$$d(w, z, a) \leq (2a_3 + a_4)d(w, z, a)$$

which is a contradiction.

Hence $w = z$

This completes the proof.

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Jour. Pure Math., Vol. 23, 2006, pp. 95-100

FIXED POINTS OF MAPPINGS AND SET-VALUED MAPPINGS IN 2-METRIC SPACE

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1. INTRODUCTION

Let (X, d) be a complete 2-metric space and let $B(X)$ be the set of all non-empty, bounded subsets of X.

The function $\delta(A, B, x)$ with A and B in $B(X)$ is defined by

$$\delta(A, B, x) = Sup\{d(a, b, x) : a \in A, b \in B\}$$

If A consists of a single point a we write

$$\delta(A, B, x) = \delta(a, B, x)$$

and if B also consists of a single point b we write

$$\delta(A, B, x) = \delta(a, b, x) = d(a, b, x)$$

Now we will prove a theorem to find the fixed point of set-valued mappings in 2-metric space.

Definition 1. An algebraic structure (X, d) is called a 2-metric space where X is a non-empty set and d is a positive real valued function defined on $X \times X \times X$ satisfying

1. $d(x, y, z) \neq 0$
2. $d(x, y, z) = 0$ if atleast two are equal
3. $d(x, y, z) = d(x, z, y) = d(y, z, x)$
4. $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z) \ \forall \ a \in X$

This concept was given by Gahler and geometrically it gives the area of triangle.

Definition 2. A sequence $\{x_n\}$ is said to Cauchy Sequence in a 2-metric space X if

$$\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$$

Definition 3. Any 2-metric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to some point $z \in X$.

Theorem. Let F be a mapping of a complete 2-metric space (X, d) into $B(X)$ and I a continuous mappings of X into itself satisfying the inequality.

$$\delta\ (Fx,\ Fy,\ a) \leq c \max \{d(Ix,\ Iy,\ a),\ (Ix,\ Fx,\ a),$$

$$\delta\ (Iy,\ Fy,\ a),\ \delta\ (Ix,\ Fy,\ a),\ \delta\ (Iy,\ Fx,\ a)\} \quad \ldots\ldots (1)$$

for all x, y in X, where $0 \leq c < 1$. If F and I commute and the range of I contains the range of F, then F and I have a unique common fixed point X and further, $Fz = \{z\}$

Proof. Let x_0 be an arbitrary point in X and choose a point y, in the set

$$Fx_0 = y_1$$

Since the range of I contains the range of F there exists a point x_1 with $Ix_1 = y_1$. In general, having chosen the point x_{n-1}. choose a point y_n in the set $Fx_{n-1} = y_n$ and then a point x_n with $Ix_n = y_n$.

Let us now suppose that the sequence of real numbers

$\{\delta(Fx_n,\ Fx_1,\ a)\}$ is unbounded.

such that

$$(1 - c)\delta(Fx_n,\ Fx_1,\ a) > c\ \delta(Fx_1,\ Fx_0,\ a)$$

and

$$\delta(Fx_n,\ Fx_1.\ a) > \max \{\delta(Fx_r,\ Fx_1,\ a) : 0 \leq r < n\} \quad \ldots\ldots (2)$$

These inequalities imply that for $r = 0, 1, 2, \ldots\ldots n$

$$c\delta(Fx_r,\ Fx_0,\ a) \leq c[\delta(Fx_r,\ Fx_1,\ a) + \delta(Fx_1,\ Fx_0,\ a) + \delta(Fx_r,\ Fx_0,\ Fx_1)]$$

$$< \delta(Fx_n,\ Fx_1,\ a)$$

Using this result in eq. (2) we get

$$\delta(Fx_n,\ Fx_1,\ a) > c * \max \{\delta(Fx_r,\ Fx_0,\ a) : 0 \leq r \leq n\} \quad \ldots\ldots (3)$$

We now prove by induction that

$$\delta(Fx_n,\ Fx_1,\ a) \leq c^k * \max \{\delta(Fx_r,\ Fx_s,\ a) : 1 \leq r,\ S \leq n\} \quad \ldots\ldots (4)$$

for $k = 1, 2, \ldots\ldots$

Using inequality (1) we have

$$\delta(Fx_n,\ Fx_1,\ a) \leq c * \max \{d(Ix_n,\ Ix_1,\ a),\ \delta(Ix_n,\ Fx_n,\ a),$$

$$\delta(Ix_1,\ Fx_1,\ a),\ \delta(Ix_n,\ Fx_1,\ a),\ \delta(Ix_1,\ Fx_n,\ a)\}$$

Using inequalities (2) and (3), this inequality reduces to

$$\delta(Fx_n, Fx_1, a) \leq c\ \delta(Fx_{n-1}, Fx_n, a)$$

Hence inequality (4) holds for $k = 1$. Now by Induction hypothesis we can assume that the result in eq. (4) holds for some k.

Then

$$\begin{aligned}\delta(Fx_n, Fx_1, a) &\leq c* \max\ \{\delta(Fx_r, Fx_s, a);\ 1 \leq r, s \leq n\}\\ &\leq c^{k+1}* \max\{\delta(Fx_{r-1}, Fx_{s-1}, a), \delta(Fx_{r-1}, Fx_r, a),\\ &\quad \delta(Fx_{s-1}, Fx_s, a), \delta(Fx_{r-1}, Fx_s, a),\\ &\quad \delta(Fx_{s-1}, Fx_r, a) : 1 \leq r, s \leq n\}\\ &\leq c^{k+1}* \max\ \{\delta(Fx_r, Fx_s, a) : 0 \leq r, s \leq n\}\end{aligned}$$

Using inequality (3), this inequality reduces to

$$\delta(Fx_n, Fx_1, a) \leq c^{k+1}* \max\ \{\delta(Fx_r, Fx_s, a) : 1 \leq r, s \leq n\}$$

and inequality (4) follows by induction.

Letting k tend to infinity in inequality (4) it now follows that

$$\delta(Fx_n, Fx_1, a) = 0$$

contradicting inequality (2). Hence

our assumption that the sequence $\{\delta(Fx_n, Fx_1, a)\}$ is unbounded is false. It follows that

$$\begin{aligned}M &= Sup\{\delta(Fx_r, Fx_s, a) : r, s = 0, 1, 2, ...\}\\ &\leq Sup\{\delta(Fx_r, Fx_s, a) + \delta(Fx_1, Fx_s, a)\\ &\quad + \delta(Fx_r, Fx_s, Fx_1) : r, s = 0, 1,.....\}\end{aligned}$$

is finite.

Now for arbitrary $\in > 0$ choose an integer N such that

$$C^N M < \in$$

Using inequality (1) for $m, n > N$ we have

$$\begin{aligned}\delta(Fx_m, Fx_n, a) &\leq c * \max\ \{\delta(Fx_{m-1}, Fx_{n-1}, a),\\ &\quad \delta(Fx_{m-1}, Fx_m, a), \delta(Fx_{n-1}, Fx_n, a),\\ &\quad \delta(Fx_{m-1}, Fx_n, a), \delta(Fx_{n-1}, Fx_m, a)\}\\ &\leq c * \max\ \{\delta(Fx_r, Fx_s, a), \delta(Fx_r, Fx_r, a),\end{aligned}$$

$$
\begin{aligned}
&\delta(Fx_{r'}, Fx_{s'}, a) : m - 1 \le r, r' \le m;\ n - 1 \le s, s' \le n\} \\
&\le c^2 * \max \{\delta(Fx_r, Fx_{r'}, a), \delta(Fx_r, Fx_{r'}, a), \\
&\delta(Fx_s, Fx_{s'}, a) : m - 2 \le r, r' \le m;\ n - 2 \le s, s' \le n\} \\
&\le c^N * \max \{\delta(Fx_r, Fx_s, a), \delta(Fx_r, Fx_{r'}, a) \\
&\delta(Fx_s, Fx_{s'}, a) : m - N \le r, r' \le m;\ n - N \le s, s' \le n\} \\
&\le c^N M \\
&< \in
\end{aligned}
$$

Thus if z_n is an arbitrary point in Fx_n for $n = 1, 2, ...$ we have

$$
\begin{aligned}
d(z_m, z_n, a) &\le \delta(z_m, Fx_n, a) \\
&\le \delta(Fx_m, Fx_n, a) \\
&< \in
\end{aligned}
$$

for $m, n > N$. It follows that the sequence $\{z_n\}$ is a Cauchy sequence in the complete 2-metric space X and so has a limit z in X. Since I is continuous, the sequence $\{Iz_n\}$ converges to Iz.

Further, there exists $\eta > 0$ such that $d(Iz_m, Iz_n, a) < \in$ whenever $d(z_m, z_n, a) < \eta$, since I is continuous, however each point z_n is chosen in Fx_n.

But there exists an integer N' such that $d(z_m, z_n, a) < \eta$ if $m, n \ge N'$ and so $d(Iz_m, I_n, a) < \in$ if $m, n \ge N'$.

Since z_n is an arbitrary point in Fx_n, it follows that $\delta(Iz_m, IFx_n, a) \le \in$ for $m, n \ge N$. In particular the point Ix_{n+1} is in Fx_n and so the sequence $\{Ix_n\}$ converges to z, the sequence $\{I^2x_n\}$ converges to Iz and

$$
\begin{aligned}
&\delta(Ix_m, Fx_n, a) < \in \\
&\delta(I^2x_m, IFx_n, a) < \in \text{ for } m, n > \max \{N, N'\}
\end{aligned}
$$

Now

$$
\begin{aligned}
d(I^2x_{n+1}, Ix_{n+1}, a) &\le \delta\ (IFx_n, Fx_n, a) \\
&= \delta(FIx_n, Fx_n, a) \\
&\le c * \max \{d(I^2x_n, Ix_n, a), \delta(I^2x_n, FIx_n, a), \\
&\qquad \delta(Ix_n, Fx_n, a), \delta(I^2x_n, Fx_n, a), \\
&\qquad \delta(Ix_n, FIx_n, a)\}
\end{aligned}
$$

$$\begin{aligned}
&\leq c * \max \{d(I^2x_n, Ix_n, a), \in, \in, \\
&\qquad d(I^2x_n, Ix_n, a) + \delta(I^2x_n, Fx_n, Ix_n) \\
&\qquad + \delta(Ix_n, Fx_n, a), d(Ix_n, I^2x_n, a) \\
&\qquad + \delta(Ix_n, FIx_n, I^2x_n) + \delta(I^2x_n, FIx_n, a)\} \\
&\leq c * \max \{d(I^2x_n, Ix_n, a), \in, \in, \\
&\qquad d(I^2x_n, Ix_n, a) + \in + \in, d(Ix_n, I^2x_n, a) + \in + \in\} \\
&\leq c * \max [d(I^2x_n, Ix_n, a) + 2\in]
\end{aligned}$$

for $n > \max \{N, N'\}$.

Letting n tend to infinity we have

$$d(Iz, z, a) \leq c[d(Iz, z, a) + 2\in]$$

and if follows that

$$Iz = z.$$

Further, since z_n is in Fx_n

$$\begin{aligned}
\delta(Fz, z_n, a) &\leq \delta(Fz, Fx_n, a) \\
&\leq c * \max \{d(Iz, Ix_n, a), \delta(Iz, Fz, a), \\
&\qquad \delta(Ix_n, Fx_n, a), \delta(Iz, Fx_n, a), \delta(Ix_n, Fz, a)\} \\
&= c * \max \{d(z, Ix_n, a), \delta(z, Fz, a), \\
&\qquad \delta(Ix_n, Fx_n, a), \delta(z, Fx_n, a), \delta(Ix_n, Fz, a)\} \\
&\leq c * \max \{\in, \delta(z, Fz, a), \in, \in, \delta(Ix_n, Fz, a)\}
\end{aligned}$$

for $n > \max \{N, N'\}$. Letting n tend to infinity we have

$$\delta(Fz, z, a) \leq c * \max \{\in, \delta(z, Fz, a)\}$$

and it follows that

$$\delta(Fz, z, a) = 0$$

Thus, z is in Fz and $Fz = \{z\}$. The point z is therefore a common fixed point of F and I.

Now suppose that F and I have a second common fixed point w. Then

$$\begin{aligned}
\delta(w, Fw, a) &\leq \delta(Fw, Fw, a) \\
&\leq c * \max \{d(Iw, Iw, a), \delta(Iw, Fw, a)\} \\
&= c * \delta(w, Fw, a)
\end{aligned}$$

So that

$$\delta(w, Fw, a) = 0$$

It follows that $Fw = \{w\}$ and so

$$d(z, w, a) = \delta(Fz, Fw, a)$$
$$\leq c\, d(z, w, a)$$

The uniqueness of z follows. This completes the proof of the theorem.

ACKNOWLEDGEMENT

Both the authors are thankful to Dr. S. S. Pagey of Institute for Excellence in Higher Education, Bhopal for his encouragement and support for the publication of the paper.

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Jour. Pure Math., Vol. 23, 2006, pp. 101-119

A NEW MULTI DIMENSIONAL INTEGRAL TRANSFORM

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ABSTRACT : In the present paper, we introduce multidimensional integral transform to present its certain interesting applications to the theory of generalized multiple hypergeometric functions of several variables including multivariable H-function of Srivastava and Panda [17, 18] We also discuss their special cases. The various operational formulas thus obtained are believed to be new. These results may be used in deriving new and known properties of special functions involved.

Key words and phrases : Multidimensional Integral Transforms, Multidimensional Laplacian Operator, Multidimensional Whittaker Transforms, Lauricella's Multiple Hypergeometric Functions, Multiple Hypergeometric Functions of Srivastava and Daoust, Multivariable H-Function of Srivastava and Panda.
2000 Mathematics Subject Classification: Primary 33C70; Secondary 44A30.

1. INTRODUCTION

Chandel [1] introduced multidimensional Laplacian operator to give integral representations of Lauricella's multiple hypergeometric functions of several variables [15]. Chandel [2] further used this operator to give integral representations of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ of Exton [12, 13]. Further Chandel and Dwivedi [4, 5] introduced multidimensional Whittaker transforms of Lauricella's multiple hypergeometric functions [15], Exton [12, 13] and generalized multiple hypergeometric function of Srivastava and Daoust [16] (also see Srivastava and Manocha [20, p. 64 (18), (19), (20)]). Recently Chandel and Kumar [6] have made applications of above multidimensional integral transforms to derive the results involving Srivastava and Panda's H-function of several complex variables [17, 18].

Very recently, Chandel and Chauhan [3] have introduced two multidimensional Laguerre transforms to present its certain applications to the theory of generalized multiple hypergeometric functions of several variables including multivariable H-function of Srivastava and Panda [17, 18].

In the present paper, making an appeal to the result due to Erdélyi et. al. [11, p. 311, eq. (28)]

$$\int_0^\infty t^{s/2-1}(1+t)^{-1/2}\left[t^{1/2}+(1+t)^{1/2}\right]^{\nu} dt = \frac{2^{1-s}\Gamma(s)\Gamma\left(\frac{1-s-\nu}{2}\right)}{\Gamma\left(\frac{1+s-\nu}{2}\right)}$$

$0 < Re(s) < 1 - Re(v)$,

we introduce a new multidimensional integral transform defined by

$$(1.1)\quad R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\{\ \} = \frac{\Gamma(\alpha_1+\ldots+\alpha_n)\Gamma(1/2+a-b+\alpha_1+\ldots+\alpha_n)2^{2a+2\alpha_1+\ldots+2\alpha_n-1}}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)\Gamma(2a+2\alpha_1+\ldots+2\alpha_n)\Gamma\left(1/2-(a+b+\alpha_1+\ldots+\alpha_n)\right)}$$

$$\int_0^\infty\ldots\int_0^\infty (x_1+\ldots+x_n)^a(1+x_1+\ldots+v_n)^{-1/2}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{2b}$$

$$x_1^{\alpha_1-1}\ldots x_n^{\alpha_n-1}\{\ \}dx_1\ldots dx_n,$$

where $0 < Re(a+\alpha_1+\ldots+\alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$.

and present its certain interesting applications to the theory of generalized multiple hypergeometric functions of several variables including multivariable H-function of Srivastava and Panda [17, 18] (also see [19], p. 251). We also discuss their special cases. The various operational formulas thus obtained are belived to be new and may be useful in deriving new and known properties of special functions involved.

2. OPERATIONAL FORMULAE

In this section, we establish the following operational results:

$$(2.1)\quad R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\{1\} = 1,$$

$$(2.2)\quad R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\left\{(x_1+\ldots+x_n)^{\xi_1+\ldots+\xi_n}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2(\eta_1+\ldots+\eta_n)}\right\}$$

$$= \frac{\Gamma(1/2+\alpha-b+\alpha_1+\ldots+\alpha_n)}{4^{\xi_1+\ldots+\xi_n}\Gamma(2a+2\alpha_1+\ldots+2\alpha_n)\Gamma\left(1/2-(a+b+\alpha_1+\ldots+\alpha_n)\right)} \times$$

$$\frac{\Gamma\left(2a+2\alpha_1+\ldots+2\alpha_n+2(\xi_1+\ldots+\xi_n)\right)\Gamma\left(1/2-(a+b+\alpha_1+\ldots+\alpha_n)+\eta_1-\xi_1+\ldots+\eta_n-\xi_n\right)}{\Gamma(1/2+a-b+\alpha_1+\ldots+\alpha_n+\xi_1+\eta_1+\ldots+\xi_n+\eta_n)}$$

where $0 < Re(a+\xi_1+\ldots+\xi_n+\alpha_1+\ldots+\alpha_n) < (1/2-b+\eta_1+\ldots+\eta_n)$ and $Re(_i) > 0$, $i = 1, \ldots, n$.

$$(2.3)\quad R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\left\{x_1^{\lambda_1}\ldots x_n^{\lambda_n}(x_1+\ldots+x_n)^{\xi_1+\ldots+\xi_n}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2(\eta_1+\ldots+\eta_n)}\right\}$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_n)\prod_{i=1}^{n}\Gamma(\alpha_i+\lambda_i)\Gamma(1/2+a+\alpha_1+\ldots+\alpha_n-b)2^{-2(\xi_1+\lambda_1+\ldots+\xi_n+\lambda_n)}}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)\Gamma\left(\sum_{i=1}^{n}(\alpha_i+\lambda_i)\Gamma(2(a+\alpha_1+\ldots+\alpha_n))\Gamma(1/2-(a+\alpha_1+\ldots+\alpha_n+b))\right)}\times$$

$$\frac{\Gamma(2(a+\alpha_1+\ldots+\alpha_n+\xi_1+\lambda_1+\ldots+\xi_n+\lambda_n))\Gamma(1/2-a-b-(\alpha_1+\lambda_1+\ldots+\alpha_n+\lambda_n)+\eta_1-\xi_1+\ldots+\eta_n-\xi_n)}{\Gamma(1/2+a-b+\alpha_1+\ldots+\alpha_n+\lambda_1+\xi_1+\eta_1+\ldots+\lambda_n+\xi_n+\eta_n)}$$

valid if $0 < Re(a+\xi_1+\ldots+\xi_n+\alpha_1+\ldots+\alpha_n+\lambda_1+\ldots+\lambda_n) < Re\,(1/2-b+\eta_1+\ldots+\eta_n)$,

$Re(\alpha_i) > 0$, $Re(\lambda_i) > 0$, $i = 1, \ldots, n$.

3. RESULTS INVOLVING SRIVASTAVA AND PANDA'S *H*-FUNCTION OF SEVERAL VARIABLES

In this section, making an appeal to (2.2) and (2.3), we derive the following results involving *H*-function of several variables due to Srivastava and Panda [17, 18]

$$(3.1)\quad R^{(a,b)}_{u_1,\ldots,\alpha_n}\left\{H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.\right.$$

$$u_1(x_1+\ldots+x_n)^{\xi_1}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\eta_1},\ldots,$$

$$\left.\left.\ldots,u_n(x_1+\ldots+x_n)^{\xi_n}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\eta_n}\right)\right\}$$

$$=\frac{\Gamma(1/2+a-b+\alpha_1+\ldots+\alpha_n)}{\Gamma(2a+2\alpha_1+\ldots+2\alpha_n)\Gamma(1/2-(a+b+\alpha_1+\ldots+\alpha_n))}$$

$$H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+2,C+1:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], & [1-2a-2\alpha_1-\ldots-2\alpha_n:2\xi_1,\ldots,2\xi_n],\\ [(c):\psi',\ldots,\psi^{(n)}], & [1/2-a+b-\alpha_1-\ldots-\alpha_n:\xi_1+\eta_1,\ldots,\xi_n+\eta_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[1/2+a+b+\alpha_1+\ldots+\alpha_n:\eta_1-\xi_1,\ldots,\eta_n-\xi_n]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{u_1}{4^{\xi_1}},\ldots,\frac{u_n}{4^{\xi_n}}\right),$$

provided that $0 < Re(a + \alpha_1 + \dots + \alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$, $\left|\arg\left(u_i / 4^{\xi_i}\right)\right| < \Delta_i \, \pi/2$,

where $\Delta_i = -\sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} - \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0$, $i = 1, \dots, n$.

$$(3.2) \quad R_{\alpha_1, \dots, \alpha_n}^{(a,b)} \left\{ H^{0,\lambda:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\dots;[B^{(n)},D^{(n)}]} \left(\begin{matrix} [(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \right. \right.$$

$$u_1 x_1^{\lambda_1}(x_1 + \dots + x_n)^{\xi_1}\left[(x_1 + \dots + x_n)^{1/2} + (1 + x_1 + \dots + x_n)^{1/2}\right]^{-2\eta_1}, \dots,$$

$$\left.\left., u_n x_n^{\lambda_n}(x_1 + \dots + x_n)^{\xi_n}\left[(x_1 + \dots + x_n)^{1/2} + (1 + x_1 + \dots + x_n)^{1/2}\right]^{-2\eta_n} \right)\right\}$$

$$= \frac{\sum_{i=1}^{n} \Gamma(\alpha_i) \prod_{i=1}^{n} \Gamma(\alpha_i + \lambda_i) \Gamma(1/2 + a - b + \alpha_1 + \dots + \alpha_n)}{\prod_{i=1}^{n} \Gamma(\alpha_i) \Gamma\left(\sum_{i=1}^{n} (\alpha_i + \lambda_i)\right) \Gamma(2a + 2\alpha_1 + \dots + 2\alpha_n) \Gamma\left(1/2 - (a + b + \alpha_1 + \dots + \alpha_n)\right)}$$

$$H^{0,\lambda+2:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A+2,C+1:[B',D'];\dots;[B^{(n)},D^{(n)}]} \left(\begin{matrix} [(a):\theta',\dots,\theta^{(n)}], \\ [(c):\psi',\dots,\psi^{(n)}], \end{matrix} \right.$$

$$\begin{matrix} [1 - 2a - 2\alpha_1 - \dots - 2\alpha_n : 2(\xi_1 + \lambda_1), \dots, 2(\xi_n + \lambda_n)], [1/2 + a + b + \alpha_1 + \dots + \alpha_n : \eta_1 - \xi_1 - \lambda_1]: \\ [1/2 - a + b - \alpha_1 - \dots - \alpha_n : \xi_1 + \eta_1 + \lambda_1, \dots, \xi_n + \eta_n + \lambda_n]: \end{matrix}$$

$$\left. \begin{matrix} [(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \; \frac{u_1}{4^{\xi_1+\lambda_1}}, \dots, \frac{u_n}{4^{\xi_n+\lambda_n}} \right),$$

valid if $0 < Re(a + \alpha_1 + \dots + \alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$, $Re(\lambda_i) > 0$

$\left|\arg\left(u_i / 4^{\xi_i+\lambda_i}\right)\right| < \Delta_i \, \pi/2$, where $\Delta_i = -\sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,\ i = 1,\dots, n.$

Special Cases of (3.2)

(i) For $\lambda_i = 0$, $i = 1, \dots, n$, (3.2) reduces to (3.1).

(ii) For $\eta_i = \xi_i$, $i = 1, \dots n$, (3.1) reduces to

$$(3.3)\quad R^{(a,b)}_{\alpha_1,\dots,\alpha_n}\left\{ H^{0,\lambda,(\mu',\nu'),\dots,(\mu^{(n)},\nu^{(n)})}_{A,C,[B',D'];\dots;[B^{(n)},D^{(n)}]}\left(\begin{matrix} [(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \right.\right.$$

$$u_1(x_1+\dots+x_n)^{\xi_1}\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^{-2\xi_1},\dots,$$

$$\left.\left.u_n(x_1+\dots+x_n)^{\xi_n}\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^{-2\xi_n}\right)\right\}$$

$$= \frac{\Gamma(1/2+a-b+\alpha_1+\dots+\alpha_n)}{\Gamma(2a+2\alpha_1+\dots+2\alpha_n)\Gamma(1/2-(a+b+\alpha_1+\dots+\alpha_n))} H^{0,\lambda+1(\mu',\nu'),\dots,(\mu^{(n)},\nu^{(n)})}_{A+1,C+1[B',D'],\dots,[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix} [(a):\theta',\dots,\theta^{(n)}],\ [1-2a-2\alpha_1-\dots-2\alpha_n : 2\xi_1,\dots,2\xi_n]: \\ [(c):\psi',\dots,\psi^{(n)}],[1/2-a+b-\alpha_1-\dots-\alpha_n : 2\xi_1,\dots,2\xi_n]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \frac{u_1}{4^{\xi_1}},\dots,\frac{u_n}{4^{\xi_n}} \right),$$

where all conditions of (3.1) are satisfied.

4. RESULTS INVOLVING MOST GENERALIZED MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

In this section, making an appeal to (2.2) and (2.3), we derive the following results involving multiple hypergeometric function of Srivastava and Daoust [16, 20]:

$$(4.1)\quad R_{\alpha_1\ldots\alpha_n}^{(a,b)}\left\{F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.\right.$$

$$u_1(x_1+\ldots+x_n)^{\xi_1}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\eta_1},\ldots,$$

$$u_n(x_1+\ldots+x_n)^{\xi_n}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\eta_n}$$

$$=F^{A+2:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], & [2a+2\alpha_1+\ldots+2\alpha_n:2\xi_1,\ldots,2\xi_n],\\ [(c):\psi',\ldots,\psi^{(n)}], & [1/2+a-b+\alpha_1+\ldots+\alpha_n:\xi_1+\eta_1,\ldots,\xi_n+\eta_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[1/2-a-b-(\alpha_1+\ldots+\alpha_n):\eta_1-\xi_1,\ldots,\eta_n-\xi_n]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{u_1}{4^{\xi_1}},\ldots,\frac{u_n}{4^{\xi_n}}\right)$$

where $0 < Re(a+\alpha_1+\ldots+\alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$,

and $1+\sum\limits_{j=1}^{C}\psi_j^{(i)}+\sum\limits_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum\limits_{j=1}^{A}\theta_j^{(i)}-\sum\limits_{j=1}^{B^{(i)}}\phi_j^{(i)} > 0$, $i = 1, 2, \ldots, n$.

$$(4.2)\quad R_{\alpha_1,\ldots,\alpha_n}^{(a,b)}\left\{F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.\right.$$

$$u_1x_1^{\lambda_1}(x_1+\ldots+x_n)^{\xi_1}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\eta_1},\ldots,$$

$$\left.\left.u_nx_n^{\lambda_n}(x_1+\ldots+x_n)^{\xi_n}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\eta_n}\right)\right\}$$

$$= F^{A+2:B'+1;\ldots;B^{(n)}+1}_{C+2:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], & [2a+2\alpha_1+\ldots+2\alpha_n:2(\xi_1+\lambda_1),\ldots,2(\xi_n+\lambda_n)],\\ [(c):\psi',\ldots,\psi^{(n)}], & [1/2+a-b+\alpha_1+\ldots+\alpha_n:\lambda_1+\xi_1+\eta_1,\ldots,\lambda_n+\xi_n+\eta_n],\end{matrix}\right.$$

$$\begin{matrix}[1/2-a-b-\alpha_1-\ldots-\alpha_n:\eta_1-\lambda_1-\xi_1,\ldots,\eta_n-\lambda_n-\xi_n]:\\ [\alpha_1+\ldots+\alpha_n:\lambda_1,\ldots,\lambda_n]:\end{matrix}$$

$$\left.\begin{matrix}[(b'):\phi'],[\alpha_1:\lambda_1];\ldots;[(b^{(n)}):\phi^{(n)}],[\alpha_n:\lambda_n]; \\ [(d'):\delta'];\ldots;\qquad [(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{u_1}{4^{\xi_1+\lambda_1}},\ldots,\frac{u_n}{4^{\xi_n+\lambda_n}}\right),$$

valid if $0 < Re(a + \alpha_1 + \ldots + \alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$, $Re(\lambda_i) > 0$

and $1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0$, $i = 1, \ldots, n$.

Special Cases of (4.2)

(i) For $\lambda_i = 0$, $i = 1, \ldots, n$. (4.2) reduces to (4.1)

(ii) For $\eta_i = \xi_i$, $i = 1, \ldots, n$, (4.1) reduces to

$$(4.3)\quad R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\left\{F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.\right.$$

$$u_1(x_1+\ldots+x_n)^{\xi_1}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\xi_1},\ldots,$$

$$\left.\left.u_n(x_1+\ldots+x_n)^{\xi_n}\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2\xi_n}\right)\right\}$$

$$= F^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], & [2a+2\alpha_1+\ldots+2\alpha_n:2\xi_1,\ldots,2\xi_n]:\\ [(c):\psi',\ldots,\psi^{(n)}], & [1/2+a-b+\alpha_1+\ldots+\alpha_n:2\xi_1,\ldots,2\xi_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{u_1}{4^{\xi_1}},\ldots,\frac{u_n}{4^{\xi_n}}\right)$$

provided that all conditions of (4.1) are satisfied.

5. SOME RESULTS OF SPECIAL INTEREST

In this section, we derive following few results of special interest:

$$(5.1)\quad R^{(a,b)}_{\alpha_1 \ldots \alpha_n}\left\{\exp\left[(u_1+\ldots+u_n)(x_1+\ldots+x_n)\Big/\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^{-2}\right]\right\}$$

$$={}_2F_2\left[\begin{matrix}a+\alpha_1+\ldots+\alpha_n, 1/2+a+\alpha_1+\ldots+\alpha_n; \\ 1/4+(a-b+\alpha_1+\ldots+\alpha_n)/2, 3/4+(a-b+\alpha_1+\ldots+\alpha_n)/2;\end{matrix}\frac{u_1+\ldots+u_n}{4}\right],$$

where $0 < Re(a+\alpha_1+\ldots+\alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$.

$$(5.2)\quad R^{(a,b)}_{\alpha_1 \ldots \alpha_n}\left\{\left[1-\frac{(u_1+\ldots+u_n)(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right]^{-\left[\frac{1+2(a-b+\alpha_1+\ldots+\alpha_n)}{4}\right]}\right\}$$

$$= {}_2F_1\left[\begin{matrix}a+\alpha_1+\ldots+\alpha_n, 1/2+a+\alpha_1+\ldots+\alpha_n; \\ \frac{3+2(a-b+\alpha_1+\ldots+\alpha_n)}{4};\end{matrix}\frac{u_1+\ldots+u_n}{4}\right]$$

$$(5.3)\quad R^{(a,b)}_{\alpha_1 \ldots \alpha_n}\left\{\left[1-\frac{(u_1+\ldots+u_n)(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right]^{-\left[\frac{3+2(a-b+\alpha_1+\ldots+\alpha_n)}{4}\right]}\right\}$$

$$= {}_2F_1\left[\begin{matrix}a+\alpha_1+\ldots+\alpha_n, 1/2+a+\alpha_1+\ldots+\alpha_n; \\ \frac{1+2(a-b+\alpha_1+\ldots+\alpha_n)}{4};\end{matrix}\frac{u_1+\ldots+u_n}{4}\right]$$

$$(5.4)\quad R^{(a,b)}_{\alpha_1 \ldots,\alpha_n}\left\{F_C^{(n)}\left(\frac{1+(2a-b+\alpha_1+\ldots+\alpha_n)}{4}, \frac{3+2(a-b+\alpha_1+\ldots+\alpha_n)}{4};\right.\right.$$

$$\left.\left. c_1,\ldots,c_n; \frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}, \ldots, \frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= F_C^{(n)}\left(a + \alpha_1 + \ldots + \alpha_n, 1/2 + a + \alpha_1 + \ldots + \alpha_n; c_1, \ldots, c_n; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right).$$

provided that $0 < Re(a + \alpha_1 + \ldots + \alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$ $|u_1|^{½} + \ldots + |u_n|^{½} < 2$.

(5.5) $$R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\left\{F_D^{(n)}\left(\frac{1 + 2(a - b + \alpha_1 + \ldots + \alpha_n)}{4}, b_1, \ldots, b_n; a + \alpha_1 + \ldots + \alpha_n;\right.\right.$$

$$\left.\left.\frac{u_1(x_1 + \ldots + x_n)}{(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}]^2}, \ldots, \frac{u_n(x_1 + \ldots + x_n)}{[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}]^2}\right)\right\}$$

$$= F_D^{(n)}\left(1/2 + a + \alpha_1 + \ldots + \alpha_n, b_1, \ldots, b_n; \frac{3 + 2(a - b + \alpha_1 + \ldots + \alpha_n}{4}; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right).$$

valid if $0 < Re(a + \alpha_1 + \ldots + \alpha_n) < 1/2 - Re(b)$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$

$\max(|u_1| + \ldots + |u_n|) < 4$.

(5.6) $$R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\left\{F_D^{(n)}\left(\frac{1 + 2(a - b + \alpha_1 + \ldots + \alpha_n)}{4}, b_1, \ldots, b_n; 1/2 + a + \alpha_1 + \ldots + \alpha_n;\right.\right.$$

$$\left.\left.;\frac{u_1(x_1 + \ldots + x_n)}{[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}]^2}, \ldots, \frac{u_n(x_1 + \ldots + x_n)}{[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}]^2}\right)\right\}$$

$$= F_D^{(n)}\left(a + \alpha_1 + \ldots + \alpha_n, b_1, \ldots, b_n; \frac{3 + 2(a - b + \alpha_1 + \ldots + \alpha_n)}{4}; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right).$$

where all conditions of (5.5) are satisfied.

(5.7) $$R^{(a,b)}_{\alpha_1,\ldots,\alpha_n}\left\{F_D^{(n)}\left(\frac{3 + 2(a - b + \alpha_1 + \ldots + \alpha_n)}{4}, b_1, \ldots, b_n; a + \alpha_1 + \ldots + \alpha_n;\right.\right.$$

$$\left.\left.\frac{u_1(x_1 + \ldots + x_n)}{[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}]^2}; \ldots, \frac{u_n(x_1 + \ldots + x_n)}{[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}]^2}\right)\right\}$$

$$= F_D^{(n)}\left(1/2 + a + \alpha_1 + \ldots + \alpha_n, b_1, \ldots, b_n; \frac{1 + 2(a - b + \alpha_1 + \ldots + \alpha_n)}{4}; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right),$$

provided that all the conditions of (5.5) are satisfied.

$$\text{(5.8)}\quad R_{\alpha_1, \ldots, \alpha_n}^{(a,b)}\left\{F_D^{(n)}\left(\frac{3 + 2(a - b + \alpha_1 + \ldots + \alpha_n)}{4}, b_1, \ldots, b_n; 1/2 + a + \alpha_1 + \ldots + \alpha_n;\right.\right.$$

$$\left.\left.\frac{u_1(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}, \ldots, \frac{u_n(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}\right)\right\}$$

$$= F_D^{(n)}\left(a + \alpha_1 + \ldots + \alpha_n, b_1, \ldots, b_n; \frac{1 + 2(a - b + \alpha_1 + \ldots + \alpha_n)}{4}; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right)$$

valid if all the conditions of (5.5) are satisfied.

6. RESULTS OF VERY SPECIAL INTEREST

In (1.1) considering $b = -[1/2 + a + \alpha_1 + \ldots + \alpha_n]$, which is valid under given conditions, we derive the following interesting results:

$$\text{(6.1)}\quad R_{\alpha_1, \ldots, \alpha_n}^{(a, -(1/2 + a + \alpha_1 + \ldots + \alpha_n))}\left\{\exp\left[\frac{(u_1 + \ldots + u_n)(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}\right]\right\}$$

$$= {}_1F_1\left[a + \alpha_1 + \ldots + \alpha_n; 1 + 3/4(a + \alpha_1 + \ldots + \alpha_n); \frac{u_1 + \ldots + u_n}{4}\right]$$

$$\text{(6.2)}\quad R_{\alpha_1, \ldots, \alpha_n}^{(a, -(1/2 + a + \alpha_1 + \ldots + \alpha_n))}\left\{F_A^{(n)}(1 + 3/4(a + \alpha_1 + \ldots + \alpha_n), b_1, \ldots, b_n; c_1, \ldots, c_n;\right.$$

$$\left.\left.\frac{u_1(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}, \ldots, \frac{u_n(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}\right)\right\}$$

$$= F_A^{(n)}\left(a + \alpha_1 + \ldots + \alpha_n, b_1, \ldots, b_n; c_1, \ldots, c_n; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right).$$

$(|u_1| + \ldots + |u_n|) < 4$, $Re(a + \alpha_1 + \ldots + \alpha_n) > 0$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$

(6.3) $R_{\alpha_1\quad,\alpha_n}^{(a.-(1/2+a+\alpha_1+\quad+\alpha_n))}\Big\{F_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;a+\alpha_1+\ldots+\alpha_n;$

$$\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\Bigg\}$$

$$= F_B^{(n)}\left(a_1,\ldots,a_n,b_1,\ldots,b_n;\ 1+3/4(a+\alpha_1+\ldots+\alpha_n);\frac{u_1}{4},\ldots,\frac{u_n}{4}\right)$$

$\max(|u_1|, \ldots,|u_n|) < 4$, $Re(a + \alpha_1 + \ldots + \alpha_n) > 0$, $Re(_i) > 0$. $i = 1, \ldots, n$.

(6.4) $R_{\alpha_1.. .\alpha_n}^{(a\ -(1/2+a+\alpha_1+\quad+\alpha_n))}\Big\{F_C^{(n)}(1+3/4(a+\alpha_1+\ldots+\alpha_n),b;c_1,\ldots,c_n;$

$$\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\Bigg\}$$

$$= F_C^{(n)}\left(a+\alpha_1+\ldots+\alpha_n,b;c_1,\ldots,c_n;\frac{u_1}{4},\ldots,\frac{u_n}{4}\right),$$

$(|u_1|^{1/2} + \ldots + |u_n|^{1/2}) < 2$, $Re(a + \alpha_1 + \ldots + \alpha_n) > 0$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$.

(6.5) $R_{\alpha_1\ldots\alpha_n}^{(a\ -(1/2+a+\alpha_1+\ldots+\alpha_n))}\Big\{F_D^{(n)}(1+3/4(a+\alpha_1+\ldots+\alpha_n),b_1,\ldots,b_n;c;$

$$\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+\left(1+x_1+\ldots+x_n\right)^{1/2}\right]^2}\right)\Bigg\}$$

$$= F_D^{(n)}\left(a+\alpha_1+\ldots+\alpha_n,b_1,\ldots,b_n;c;\frac{u_1}{4},\ldots,\frac{u_n}{4}\right),$$

$\max(|u_1|,\ldots,|u_n|) < 4$, $Re(a + \alpha_1 + \ldots + \alpha_n) > 0$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$.

(6.6) $R_{\alpha_1\ldots\alpha_n}^{(a.-(1/2+a+\alpha_1+\quad+\alpha_n))}\Big\{F_D^{(n)}(a,b_1,\ldots,b_n;a+\alpha_1+\ldots+\alpha_n;$

$$\left.\left.\frac{u_1(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2},\dots,\frac{u_n(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$=F_D^{(n)}\left(a,b_1,\dots,b_n;1+3/4(a+\alpha_1+\dots+\alpha_n);\frac{u_1}{4},\dots,\frac{u_n}{4}\right)$$

$\max(|u_1|,\dots,|u_n|)<4$, $Re(a+\alpha_1+\dots+\alpha_n)>0$. $Re(\alpha_i)>0$, $i=1,\dots,n$.

(6.7) $$R^{(a,-(1/2+a+\alpha_1+\dots+\alpha_n))}_{u_1\dots u_n}\left\{F_D^{(n)}\left(1+3/4(a+\alpha_1+\dots+\alpha_n),b_1,\dots,b_n;a+\alpha_1+\dots+\alpha_n;\right.\right.$$

$$\left.\left.\frac{u_1(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2},\dots,\frac{u_n(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$=\prod_{i=1}^{n}\left(1-\frac{u_i}{4}\right)^{-b_i},$$

$\max(|u_1|,\dots,|u_n|)<4$, $Re(a+\alpha_1+\dots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\dots,n$.

(6.8) $$R^{(a,-(1/2+a+\alpha_1+\dots+\alpha_n))}_{u_1\dots u_n}\left\{\psi_2^{(n)}\left(1+3/4(a+\alpha_1+\dots+\alpha_n),c_1,\dots,c_n;\right.\right.$$

$$\left.\left.\frac{u_1(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2},\dots,\frac{u_n(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$=\psi_2^{(n)}\left(a+\alpha_1+\dots+\alpha_n;c_1,\dots,c_n;\frac{u_1}{4},\dots,\frac{u_n}{4}\right)$$

$Re(a+\alpha_1+\dots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\dots,n$

(6.9) $$R^{(a,-(1/2+a+\alpha_1+\dots+\alpha_n))}_{u_1\dots u_n}\left\{\phi_2^{(n)}\left(c_1,\dots,c_n;a+\alpha_1+\dots+\alpha_n;\right.\right.$$

$$\left.\left.\frac{u_1(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2},\dots,\frac{u_n(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= \phi_2^{(n)}\left(c_1,\ldots,c_n;1+3/4(a+\alpha_1+\ldots+\alpha_n);\frac{u_1}{4},\ldots,\frac{u_n}{4}\right),$$

$Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$.

(6.10) $$R^{(a,-(1/2+a+\alpha_1+\quad+u_n))}_{u_1,\ ,u_n}\left\{{}^{(k)}_{(1)}E^{(n)}_{C}(b,b',1+3/4(a+\alpha_1+\ldots+\alpha_n);c_1,\ldots,c_n;\right.$$

$$\left.\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(1)}E^{(n)}_{C}\left(b,b',a+\alpha_1+\ldots+\alpha_n;c_1,\ldots,c_n;\frac{u_1}{4},\ldots,\frac{u_n}{4}\right),$$

$Re(a+\alpha_1+\ldots+\alpha_n)>0$, $|u_i|<4r_i$, $\left(\sqrt{r_1}+\ldots+\sqrt{r_k}\right)^2+\left(\sqrt{r_{k+1}}+\ldots+\sqrt{r_n}\right)^2=1$,

$Re(\alpha_i)>0$,

$i=1,\ldots,n$; where r_i are associated radii of convergence of sereis ${}^{(k)}_{(1)}E^{(n)}_{C}$.

(6.11) $$R^{(a,-(1/2+a+\alpha_1+\quad+\alpha_n))}_{\alpha_1,\ ,\alpha_n}\left\{{}^{(k)}_{(1)}E^{(n)}_{D}(1+3/4(a+\alpha_1+\ ..+\alpha_n);b_1,\ldots,b_n,c,c';\right.$$

$$\left.\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(1)}E^{(n)}_{D}\left(a+\alpha_1+\ldots+\alpha_n,b_1,\ldots b_n;c,c';\frac{u_1}{4},\ldots,\frac{u_n}{4}\right),$$

where $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, if $|u_i|<4r_i$ then $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_n$, $r_k+r_n=1$, where r_i are the associated radii of convergence of seres ${}^{(k)}_{(1)}E^{(n)}_{D}$; $i=1,\ldots,n$.

(6.12) $$R^{(a,-(1/2+a+\alpha_1+\quad+\alpha_n))}_{\alpha_1,\ ,\alpha_n}\left\{{}^{(k)}_{(2)}E^{(n)}_{D}(b,b';c_1,\ldots,c_n;a+\alpha_1+\ldots+\alpha_n;\right.$$

$$\left.\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(2)}E_D^{(n)}\left(b, b', c_1, \ldots, c_n; 1 + 3/4(a + \alpha_1 + \ldots + \alpha_n); \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right)$$

where $Re(a + \alpha_1 + \ldots + \alpha_n) > 0$, $Re(\alpha_i) > 0$, $|u_i| < 4r_i$, $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_n$, $r_k + r_n = r_k r_n$; $i = 1, \ldots, n$.

(6.13) $$R_{\alpha_1, \ldots, \alpha_n}^{(a, -(1/2 + a + \alpha_1 + \ldots + \alpha_n))}\left\{{}^{(k)}F_{AC}^{(n)}\left(1 + 3/4(a + \alpha_1 + \ldots + \alpha_n), b, b_{k+1}, \ldots, b_n, c_1, \ldots, c_n; \right.\right.$$

$$\left.\left.\frac{u_1(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}, \ldots, \frac{u_n(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}F_{AC}^{(n)}\left(a + \alpha_1 + \ldots + \alpha_n, b, b_{k+1}, \ldots, b_n; c_1, \ldots, c_n; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right),$$

$Re(a + \alpha_1 + \ldots + \alpha_n) > 0$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$; $(|u_1|^{1/2} + \ldots + |u_k|^{1/2})^2 + |u_{k+1}| + \ldots + |u_n| < 4$.

(6.14) $$R_{\alpha_1, \ldots, \alpha_n}^{(a, -(1/2 + a + \alpha_1 + \ldots + \alpha_n))}\left\{{}^{(k)}F_{AD}^{(n)}\left(1 + 3/4(a + \alpha_1 + \ldots + \alpha_n), b_1, \ldots, b_n; c, c_{k+1}, \ldots, c_n; \right.\right.$$

$$\left.\left.\frac{u_1(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}, \ldots, \frac{u_n(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}F_{AD}^{(n)}\left(a + \alpha_1 + \ldots + \alpha_n, b_1, \ldots, b_n; c, c_{k+1}, \ldots, c_n; \frac{u_1}{4}, \ldots, \frac{u_n}{4}\right),$$

where $Re(a + \alpha_1 + \ldots + \alpha_n) > 0$, $Re(\alpha_i) > 0$, $i = 1, \ldots, n$; $max\ (|u_1| + \ldots + |u_k|) + |u_{k+1}| + \ldots + |u_n| < 4$.

(6.15) $$R_{\alpha_1, \ldots, \alpha_n}^{(a, -(1/2 + a + \alpha_1 + \ldots + \alpha_n))}\left\{{}^{(k)}F_{BD}^{(n)}\left(b, b_{k+1}, \ldots, b_n, c_1, \ldots, c_n; a + \alpha_1 + \ldots + \alpha_n; \right.\right.$$

$$\left.\left.\frac{u_1(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}, \ldots, \frac{u_n(x_1 + \ldots + x_n)}{\left[(x_1 + \ldots + x_n)^{1/2} + (1 + x_1 + \ldots + x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}F_{BD}^{(n)}\left(b. b_{k+1},\ldots,b_n;c_1,\ldots,c_n;\ 1+3/4(a+\alpha_1+\ldots+\alpha_n);\frac{u_1}{4},\ldots,\frac{u_n}{4}\right)$$

valid if $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$; $max\,(|u_1|+\ldots+|u_n|)<4$.

(6.16) $$R_{\alpha_1,\ldots,\alpha_n}^{(a,-(1/2+a+\alpha_1+..+\alpha_n))}\Bigg\{{}^{(k)}F_{CD}^{(n)}\Big(1+3/4(a+\alpha_1+\ldots+\alpha_n),b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;$$

$$\left.\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}F_{CD}^{(n)}\left(a+\alpha_1+\ldots+\alpha_n,b,b_1,\ldots,b_k;c,c_{k+1},\ldots,c_n;\frac{u_1}{4},\ldots,\frac{u_n}{4}\right)$$

provided that $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$.

(6.17) $$R_{\alpha_1,\ldots,\alpha_n}^{(a,-(1/2+a+\alpha_1+..+\alpha_n))}\Bigg\{{}^{(k)}_{(1)}\phi_{AC}^{(n)}\Big(1+3/4(a+\alpha_1+\ldots+\alpha_n),b;c_1,\ldots,c_n;$$

$$\left.\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(1)}\phi_{AC}^{(n)}\left(a+\alpha_1+\ldots+\alpha_n,b;c_1,\ldots,c_n;\frac{u_1}{4},\ldots,\frac{u_n}{4}\right)$$

valid if $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$.

(6.18) $$R_{\alpha_1,\ldots,\alpha_n}^{(a,-(1/2+a+\alpha_1+..+\alpha_n))}\Bigg\{{}^{(k)}_{(2)}\phi_{AC}^{(n)}\Big(1+3/4(a+\alpha_1+\ldots+\alpha_n),b_{k+1},\ldots,b_n;c_1,\ldots,c_n;$$

$$\left.\left.\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(2)}\phi_{AC}^{(n)}\left(a+\alpha_1+\ldots+\alpha_n,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;\frac{u_1}{4},\ldots,\frac{u_n}{4}\right)$$

provided that $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$.

(6.19) $$R^{(a,-(1/2+a+\alpha_1+\ldots+\alpha_n))}_{\alpha_1,\ldots,\alpha_n}\left\{{}^{(k)}_{(1)}\phi^{(n)}_{AD}\left(1+3/4(a+\alpha_1+\ldots+\alpha_n),b_1,\ldots,b_n;c;\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$={}^{(k)}_{(1)}\phi^{(n)}_{AD}\left(a+\alpha_1+\ldots+\alpha_n+b_1,\ldots,b_n;c;\frac{u_1}{4},\ldots,\frac{u_n}{4}\right),$$

where $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$.

(6.20) $$R^{(a,-(1/2+a+\alpha_1+\ldots+\alpha_n))}_{\alpha_1,\ldots,\alpha_n}\left\{{}^{(k)}_{(1)}\phi^{(n)}_{BD}\left(b,b_1,\ldots,b_n;a+\alpha_1+\ldots+\alpha_n;\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$={}^{(k)}_{(1)}\phi^{(n)}_{BD}\left(b,b_1,\ldots,b_n;1+3/4(a+\alpha_1+\ldots+\alpha_n);\frac{u_1}{4},\ldots,\frac{u_n}{4}\right),$$

valid if $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$.

(6.21) $$R^{(a,-(1/2+a+\alpha_1+\ldots+\alpha_n))}_{\alpha_1,\ldots,\alpha_n}\left\{{}^{(k)}_{(2)}\phi^{(n)}_{BD}\left(b_{k+1},\ldots,b_n,c_1,\ldots,c_n;a+\alpha_1+\ldots+\alpha_n;\frac{u_1(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2},\ldots,\frac{u_n(x_1+\ldots+x_n)}{\left[(x_1+\ldots+x_n)^{1/2}+(1+x_1+\ldots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$={}^{(k)}_{(2)}\phi^{(n)}_{BD}\left(b_{k+1},\ldots,b_n;c_1,\ldots,c_n;1+3/4(a+\alpha_1+\ldots+\alpha_n);\frac{u_1}{4},\ldots,\frac{u_n}{4}\right)$$

provided that $Re(a+\alpha_1+\ldots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\ldots,n$.

(6.22) $$R^{(a,-(1/2+a+\alpha_1+\ldots+\alpha_n))}_{\alpha_1,\ldots,\alpha_n}\left\{{}^{(k)}_{(1)}\phi^{(n)}_{CD}(1+3/4(a+\alpha_1+\ldots+\alpha_n),b;c,c_{k+1},\ldots,c_n;\right.$$

$$\left.\left.\frac{u_1(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2},\dots,\frac{u_n(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{CD}\left(a+\alpha_1+\dots+\alpha_n, b; c, c_{k+1},\dots,c_n; \frac{u_1}{4},\dots,\frac{u_n}{4}\right)$$

valid if $Re(a+\alpha_1+\dots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\dots,n$.

(6.23) $R^{(a,-(1/2+a+\alpha_1+\dots+\alpha_n))}_{\alpha_1,\dots,\alpha_n}\left\{{}^{(k)}_{(1)}\phi^{(n)}_{CD}(1+3/4(a+\alpha_1+\dots+\alpha_n), b_1,\dots,b_n; c, c_{k+1},\dots,c_n;\right.$

$$\left.\left.\frac{u_1(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2},\dots,\frac{u_n(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(2)}\phi^{(n)}_{CD}\left(a+\alpha_1+\dots+\alpha_n, b; c, c_{k+1},\dots,c_n; \frac{u_1}{4},\dots,\frac{u_n}{4}\right)$$

where $Re(a+\alpha_1+\dots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\dots,n$.

(6.24) $R^{(a,-(1/2+a+\alpha_1+\dots+\alpha_n))}_{\alpha_1,\dots,\alpha_n}\left\{{}^{(k)}_{(4)}\phi^{(n)}_{CD}(1+3/4(a+\alpha_1+\dots+\alpha_n), b, b_1,\dots,b_k; c;\right.$

$$\left.\left.\frac{u_1(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2},\dots,\frac{u_n(x_1+\dots+x_n)}{\left[(x_1+\dots+x_n)^{1/2}+(1+x_1+\dots+x_n)^{1/2}\right]^2}\right)\right\}$$

$$= {}^{(k)}_{(4)}\phi^{(n)}_{CD}\left(a+\alpha_1+\dots+\alpha_n, b, b_1,\dots b_k; c; \frac{u_1}{4},\dots,\frac{u_n}{4}\right),$$

provided that $Re(a+\alpha_1+\dots+\alpha_n)>0$, $Re(\alpha_i)>0$, $i=1,\dots,n$.

Here $F^{(n)}_A, F^{(n)}_B, F^{(n)}_C, F^{(n)}_D$ are multiple hypergeometric functions of Lauricella [15] and $\phi^{(n)}_2, \psi^{(n)}_2$ are their confluent forms. ${}^{(k)}_{(1)}E^{(n)}_D$, ${}^{(k)}_{(2)}E^{(n)}_D$ are Exton's multiple hypergeometric functions [12, 13] related to Lauricella's $F^{(n)}_D$, while ${}^{(k)}_{(1)}E^{(n)}_C$ is Chandel's multiple hypergeometric function [2] related to Lauricella's $F^{(n)}_C$.

${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ are intermediate Lauricella's functions introduced by Chandel and Gupta [6] with their confluent forms ${}^{(k)}_{(1)}\phi^{(n)}_{AC}$, ${}^{(k)}_{(2)}\phi^{(n)}_{AC}$, ${}^{(k)}_{(1)}\phi^{(n)}_{AD}$, ${}^{(k)}_{(1)}\phi^{(n)}_{BD}$, ${}^{(k)}_{(2)}\phi^{(n)}_{BD}$. Further ${}^{(k)}_{(1)}\phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\phi^{(n)}_{CD}$, ${}^{(k)}_{(3)}\phi^{(n)}_{CD}$ and ${}^{(k)}_{(4)}\phi^{(n)}_{CD}$ are confluent forms of Karlsson's multiple hypergeometric function ${}^{(k)}F^{(n)}_{CD}$ [14] introduced and studied by Chandel and Vishwakarma [7, 8, 9].

Acknowledgement:

The authors are thankful to Professor H. M. Srivastava (Victoria) for his suggestions to bring the paper in its present form.

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Jour. Pure Math., Vol. 23, 2006, pp. 121-134

SOME PROPERTIES OF A SEMI-SYMMETRIC NON-METRIC CONNECTION IN A RIEMANNIAN MANIFOLD

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ABSTRACT : Agashe and Chafle in 1992 introduced the idea of semi-symmetric non-metric connection on a Riemannian manifold and this was further developed by many geometers. Recently Prasad and Verma (2004) defined a semi-symmetric non-metric connection $\overline{\nabla}$ on a Riemannian manifold which was generalization of the notion of semi-symmetric non-metric connection introduced by Agashe and Chafle. They obtained some properties of curvature tensor and projective curvature tensor with respect to semi-symmetric non-metric connection $\overline{\nabla}$.

In this paper we obtain first and second Bianchi identities associated with semi-symmetric non-metric connection $\overline{\nabla}$. A necessary and sufficient condition has been deduced for the Ricci tensor $\overline{\nabla}$ to be skew-symmetric under certain conditions. We also define conformal curvature tensor and concircular curvature tensor with respect to semi-symmetric non-metric connection and obtain the relations connecting it and conformal curvature tensor and concircular curvature tensor with respect to Levi-Civita connection.

1. INTRODUCTION

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ endowed with a Riemannian metric g and let ∇ be the Levi-Civita connection on (M^n, g).

A linear connection $\overline{\nabla}$ defined on (M^n, g) is said to be semi-symmetric (Friedmann and Schouten, 1924) if its torsion tensor $\overline{T}$ is of the form.

$$\overline{T}(X, Y) = u(Y)X - u(X)Y + a(X)Y - a(Y)X \tag{1.1}$$

where u and a are 1-forms associated with vector fields U and A on M^n by

$$u(X) = g(X, U) \tag{1.2}$$

and

$$a(X) = g(X, A) \tag{1.3}$$

for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n.

A semi-symmetric connection $\bar{\nabla}$ is called a semi-symmetric non-metric connection (Prasad and Verma, 2004) if it satisfies

$$(\bar{\nabla} \times g)(Y, Z) = -u(Y)g(X, Z) - u(Z)\, g(X, Y) - 2a(X)\, g(Y, Z). \tag{1.4}$$

A relation between the semi-symmetric non-metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on (M^n, g) has been obtained by Prasad and Verma (2004) which is giving by

$$\bar{\nabla}_X Y = \nabla_X Y + u(Y)X + a(X)Y. \tag{1.5}$$

Further, if $\bar{R}$ and R denote the curvature tensor of $\bar{\nabla}$ and ∇ respectively, then it is known (Prasad and Verma, 2004) that

$$\bar{R}(X, Y)Z = R(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X + da(X, Y)Z \tag{1.6}$$

where α is a tensor field of the type (0, 2) defined on M^n by

$$\alpha(X, Y) = (\nabla_X u)Y - u(X)\, u(Y) = (\bar{\nabla}_X u)Y + a(X)u(Y). \tag{1.7}$$

From (1.7), we have

$$\alpha(X, Y) - \alpha(Y, X) = du(X, Y). \tag{1.8}$$

Further, from (1.6), we have

$$\bar{R}ic(Y, Z) = Ric(Y, Z) - (n - 1)\alpha(Y, Z) - da(Y, Z) \tag{1.9}$$

where $\bar{R}ic(Y, Z)$ and $Ric(Y, Z)$ denote the Ricci tensor of $\bar{\nabla}$ and ∇ respectively.

From (1.9), we obtain relation between the scalar curvature of (M^n, g) with respect to Riemannian connection ∇ and the semi-symmetric non-metric connection $\bar{\nabla}$ which is given by

$$\bar{r} = r - (n - 1)p \tag{1.10}$$

where $p = \text{trace}\ \alpha$ and $r = \sum_{i=1}^{n} Ric\,(E_i, E_i)$ is the scalar curvature of M^n with respect to Riemannian connection.

2. PROPERTIES OF CURVATURE TENSOR OF M^n WITH RESPECT TO SEMI-SYMMETRIC NON-METRIC CONNECTION

Covariant differentiation of torsion tensor $\bar{T}$ is given by

$$\begin{aligned}(\bar{\nabla}_X \bar{T})(Y, Z) &= \bar{\nabla}_X(\bar{T}(Y, Z) - \bar{T}(\bar{\nabla}_X Y, Z) - \bar{T}(Y, \bar{\nabla}_X Z)\\ &= ((\bar{\nabla}_X u)Z)Y - ((\bar{\nabla}_X u)Y)Z + ((\bar{\nabla}_X a)Y)Z - ((\bar{\nabla}_X a)Z)Y\\ &= \alpha(X, Z)Y - \alpha(X, Y)Z + \beta(X, Y)Z - \beta(X, Z)Y\end{aligned} \tag{2.1}$$

where β is a tensor of type (0, 2) given by

$$\beta(X, Y) = (\nabla_X a)Y - a(X)a(Y) = (\bar{\nabla}_X a)Y + a(X)u(Y). \tag{2.2}$$

From (2.2), we have

$$\beta(X,Y) - \beta(Y, X) = da(X, Y). \tag{2.3}$$

Thus a tensor β is symmetric if and only if 1-form a is closed.

Further, we define $\bar{T}(X, Y, Z) = g(\bar{T}(X, Y), Z)$. (2.4)

From (1.1) and (2.4), we have

$$\bar{T}(X, Y, Z) + \bar{T}(Y, Z, X) + \bar{T}(Z, X, Y) = 0 \tag{2.5}$$

From (1.8), (2.1) and (2.3), we have

$$\begin{aligned}&(\bar{\nabla}_X\bar{T})(Y,Z) + (\bar{\nabla}_Y\bar{T})(Z, X) + (\bar{\nabla}_Z\bar{T})(X, Y)\\ &= du(Y, X)Z + du(X, Z)Y + du(Z, Y)X + da(X, Y)Z + da(Y, Z)X + da(Z, X)Y.\end{aligned} \tag{2.6}$$

From (1.1), we have

$$\begin{aligned}&\bar{T}(\bar{T}(X, Y),Z) + \bar{T}(\bar{T}(Y, Z), X) + \bar{T}(\bar{T}(Z, X), Y)\\ &= \bar{T}(u(Y)X - u(X)Y + a(X)Y - a(Y)X,Z) + \bar{T}(u(Z)Y - u(Y)Z + a(Y)Z - a(Z)Y,X)\\ &+ \bar{T}(u(X)Z - u(Z)X + a(Z)X - a(X)Z, Y) = 0.\end{aligned} \tag{2.7}$$

From (2.5). (2.6) and (2.7), we can state the theorem.

Theorem 2.1. The torsion tensor $\bar{T}$ with respect to the semi-symmetric non-metric connection satisfies following algebraic properties

(i) ${}'\bar{T}(X, Y, Z) + {}'\bar{T}(Y, Z, X) + {}'\bar{T}(Z, X, Y) = 0$

(ii) $\bar{T}(\bar{T}(X, Y), Z) + \bar{T}(\bar{T}(Y, Z), X) + \bar{T}(\bar{T}(Z, X), Y) = 0$

(iii) $(\bar{\nabla}_X \bar{T})(Y, Z) + (\bar{\nabla}_Y \bar{T})(Z, X) + (\bar{\nabla}_Z \bar{T})(X, Y)$
$= du(Y, X)Z + du(X, Z)Y + du(Z, Y)X + da(X, Y)Z + da(Y, Z)X + da(Z, X)Y.$

Bianchi first identity for a linear connection on M^n is given by (Sinha, 1982)

$$\begin{aligned}&\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y\\ &= \bar{T}(\bar{T}(X, Y), Z) + \bar{T}(\bar{T}(Y, Z), X) + \bar{T}(\bar{T}(Z, X), Y) + (\bar{\nabla}_X \bar{T})(Y, Z)\\ &\qquad + (\bar{\nabla}_Y \bar{T})(Z, X) + (\bar{\nabla}_Z \bar{T})(X, Y).\end{aligned}$$

Using (2.6) and (2.7) in above expression, we get

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y$$
$$= du(Y, X)Z + du(X, Z)Y + du(Z, Y)X + da(X, Y)Z + da(Y, Z)X + da(Z, X)Y. \tag{2.8}$$

We call (2.8) as the first Bianchi identity with respect to semi-symmetric non-metric connection $\bar{\nabla}$.

Remark 1. This identity has been obtained by Prasad and Verma in their paper 2004 by another way.

Remark 2. In particular, if $a = 0$, then from (2.8), we get

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y$$
$$= du(Y, X)Z + du(X, Z)Y + du(Z, Y)X. \tag{2.9}$$

i.e. we get the same expression for the curvature tensor $\bar{R}$ of the manifold with respect to $\bar{\nabla}$ as was obtained by Agashe and Chafle (1992) in their paper by another way.

Let $'\bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W)$ (2.10)

and $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$. (2.11)

In view of (1.6), (2.10) and (2.11), we get

$$'\bar{R}(X, Y, Z, W) + {}'\bar{R}(Y, X, Z, W) = 0. \tag{2.12}$$

We have

$$\bar{R}(\bar{T}(X, Y), Z) + \bar{R}(\bar{T}(Y, Z), X) + \bar{R}(\bar{T}(Z, X), Y)$$
$$= -2u(Z)\bar{R}(X, Y) - 2u(X)\bar{R}(Y, Z) - 2u(Y)\bar{R}(Z, X) + 2a(Z)\bar{R}(X, Y)$$
$$+ 2a(X)\bar{R}(Y, Z) + 2a(Y)\bar{R}(Z, X). \tag{2.13}$$

Bianchi second identity for a linear connection on M^n is given by (Sinha, 1982)

$$(\bar{\nabla}_X\bar{R})(Y, Z) + (\bar{\nabla}_Y\bar{R})(Z, X) + (\bar{\nabla}_Z\bar{R})(X, Y)$$
$$= -\bar{R}(\bar{T}(X, Y), Z) - \bar{R}(\bar{T}(Y, Z), X) - \bar{R}(\bar{T}(Z, X), Y).$$

Using (2.13) in above expression we get

$$(\bar{\nabla}_X \bar{R})(Y, Z) + (\bar{\nabla}_Y \bar{R})(Z, X) + (\bar{\nabla}_Z \bar{R})(X, Y)$$
$$= 2u(Z)\bar{R}(X, Y) + 2u(X)\bar{R}(Y, Z) + 2u(Y)\bar{R}(Z, X) - 2a(Z)\bar{R}(X, Y)$$
$$- 2a(X)\bar{R}(Y, Z) - 2a(Y)\bar{R}(Z, X). \tag{2.14}$$

We call (2.14) as a second Bianchi identity with respect to semi-symmetric non-metric connection $\overline{\nabla}$.

Remark. In particular, if a = 0, then from (2.14) we get same expression as was obtained by Agashe and Chafle in their paper 1992 by another way.

If Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then from (1.9), we get

$$\alpha(Y, Z) = \frac{1}{n-1}[\mathrm{Ric}(Y, Z) - da(Y, Z)]. \tag{2.15}$$

From (2.15), we have

$$\alpha(Z, Y) = \frac{1}{n-1}[\mathrm{Ric}(Z, Y) - da(Z, Y)]. \tag{2.16}$$

From (1.8), (2.15) and (2.16), we get

$$du(Y, Z) = -\frac{2}{n-1}da(Y, Z) \tag{2.17}$$

From (2.6) and (2.17), we get

$$\begin{aligned}&(\overline{\nabla}_X \overline{T})(Y, Z) + (\overline{\nabla}_Y \overline{T})(Z, X) + (\overline{\nabla}_Z \overline{T})(X, Y)\\ &= \frac{(n+1)}{(n-1)}[da(X, Y)Z + da(Y, Z)X + da(Z, X)Y].\end{aligned} \tag{2.18}$$

Hence we can state the following theorem.

Theorem 2.2. If Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then

$$\begin{aligned}&(\overline{\nabla}_X \overline{T})(Y, Z) + (\overline{\nabla}_Y \overline{T})(Z, X) + (\overline{\nabla}_Z \overline{T})(X, Y)\\ &= \frac{n+1}{n-1}[da(X, Y)Z + da(Y, Z)X + da(Z, X)Y].\end{aligned}$$

Remark. In particular, if a = 0, then from (2.18) we get the same expression as was obtained by Agashe and Chafle (1992) in their paper.

From (1.6), (2.1) and (2.15), we can state the following theorem.

Theorem 2.3. Let M^n be a Riemannian manifold with semi-symmetric non-metric connection. If Ricci tensor with respect to semi-symmetric non-metric connection vanishes, then

$$\begin{aligned}R(X, Y)Z &= \overline{R}(X, Y)Z + (\overline{\nabla}_Z \overline{T})(X, Y) + \beta(Z, Y)X\\ &- \beta(Z, X)Y - \frac{2}{n-1}[da(Y, Z)X + da(Z, X)Y] - da(X, Y)Z.\end{aligned} \tag{2.19}$$

Remark. In particular, if $a = 0$, then from (2.19), we get

$R(X, Y)Z = \bar{R}(X, Y)Z + (\bar{\nabla}_Z \bar{T})(X, Y)$ which is same as was obtained by Agashe and Chafle (1992).

Let $\bar{R}(X, Y)Z = 0$, (2.20)

which gives

$$\bar{R}ic(Y, Z) = 0 \quad \text{on } M^n. \tag{2.21}$$

From (2.19) and (2.20), we have

$$R(X, Y)Z = (\bar{\nabla}_Z \bar{T})(X, Y) + \beta(Z, Y)X - \beta(Z, X)Y$$

$$- \frac{2}{n-1}[da(Y, Z)X + da(Z, X)Y] - da(X, Y)Z. \tag{2.22}$$

Thus, we have

Theorem 2.4. If M^n is a Riemannian manifold with vanishing curvature tensor with respect to semi-symmetric non-metric connection, then M^n is flat if and only if

$$(\bar{\nabla}_Z \bar{T})(X, Y) = \beta(Z, X)Y - \beta(Z, Y)X$$

$$+ \frac{2}{n-1}[da(Y, Z)X + da(Z, X)Y] + da(X, Y)Z. \tag{2.23}$$

Remark. In particular, if $a = 0$, then from (2.23), we get $(\bar{\nabla}_Z \bar{T})(X, Y) = 0$ which is same as was obtained by Agashe and Chafle (1992) in their paper.

A Riemannian manifold M^n is a group manifold with respect to semi-symmetric non-metric connection if

$$\bar{R}(X, Y)Z = 0$$

and $(\bar{\nabla}_Z \bar{T})(X, Y) = 0$ on M^n. (2.24)

From (2.19) and (2.24) we can state following theorem.

Theorem 2.5. If a Riemannian manifold M^n is a group manifold with respect to semi-symmetric non-metric connection, then

$$R(X, Y)Z = \beta(Z, Y)X - \beta(Z, X)Y - \frac{2}{n-1}[da(Y, Z)X + da(Z, X)Y] - da(X, Y)Z. \tag{2.25}$$

Remark. In particular, if $a = 0$, then from (2.25), we get $R(X, Y)Z = 0$, then M^n is flat and consequently M^n is projectively flat and conformally flat which is the result as was obtained by Agashe and Chafle (1992) in their paper.

A Riemannian manifold M^n is called an Einstein manifold with respect to Riemannian connection if

$$\mathrm{Ric}(X, Y) = \frac{r}{n}g(X, Y). \tag{2.26}$$

Analogous to this definition, we define that a Riemannian manifold M^n with respect to semi-symmetric non-metric connection is called an Einstein manifold if

$$\overline{R}ic(X, Y) = \frac{\overline{r}}{n}g(X, Y) \tag{2.27}$$

From (1.9) and (1.10) we have

$$\overline{R}ic(X, Y) - \frac{\overline{r}}{n}g(X, Y) = \mathrm{Ric}(X, Y) - \frac{r}{n}g(X, Y)$$

$$- (n - 1)\alpha(X, Y) - da(X, Y) + \frac{p(n-1)}{n}g(X, Y). \tag{2.28}$$

If $(n - 1)\alpha(X, Y) + da(X, Y) = \frac{p(n-1)}{n}g(X, Y)$ (2.29)

Then from (2.28), we get

$$\overline{R}ic(X, Y) - \frac{\overline{r}}{n}g(X, Y) = \mathrm{Ric}(X, Y) - \frac{r}{n}g(X, Y) \tag{2.30}$$

From (2.30), we can state the following theorem.

Theorem 2.6. If a Riemannian manifold M^n admits a semi-symmetric non-metric connection and if the relation (2.29) holds, then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\overline{\nabla}$.

3. KILLING VECTOR FIELD

In this section we shall consider

$$(\nabla_Y u)X + (\nabla_X u)Y = 0. \tag{3.1}$$

Then U is called a killing vector field.

Taking covariant differentiation of (1.2) along Y, we get

$$g(X, \nabla_Y U) = (\nabla_Y u)X. \tag{3.2}$$

From (3.2), we have

$$g(Y, \nabla_X U) = (\nabla_X u)Y \tag{3.3}$$

From (3.1), (3.2) and (3.3), we have

$$g(X, \nabla_Y U) + g(Y, \nabla_X U) = 0. \tag{3.4}$$

Thus the killing vector field U satisfies (3.4).

From (1.7) and (1.9), we get

$$\bar{R}ic(X, Y) = Ric(X, Y) - (n - 1)[(\nabla_X u)Y - u(X)u(Y)] - da(X, Y). \tag{3.5}$$

In view of (3.5), we have

$$\bar{R}ic(Y, X) = Ric(Y, X) - (n-1)[(\nabla_Y u)X - u(Y)u(X)] - da(Y, X). \tag{3.6}$$

From (3.1), (3.5) and (3.6), we get

$$\bar{R}ic(X, Y) + \bar{R}ic(Y, X) = 2Ric(X, Y) + 2(n - 1)u(X)\, u(Y). \tag{3.7}$$

If $\bar{R}ic(X, Y)$ is skew-symmetric the left hand side of (3.7) vanishes, and we get

$$Ric(X, Y) = (1 - n)u(X)u(Y). \tag{3.8}$$

On the other hand if Ric(X, Y) is given by (3.8), then from (3.7), we get

$$\bar{R}ic(X, Y) + \bar{R}ic(Y, X) = 0.$$

Hence we can state the following theorem.

Theorem 3.1. If a Riemannian manifold admits a semi-symmetric non-metric connection $\bar{\nabla}$ with U as a killing vector field, then a necessary and sufficient condition for the Ricci tensor of $\bar{\nabla}$ to be skew-symmetric is that the Ricci tensor of Levi-Civita connection ∇ is given by (3.8).

Also, if Ric(X. Y) = 0, then from (3.8), we get

$$u(X)u(Y) = 0 \quad \text{for } n \geq 2$$

which gives on contraction

$$u(U) = 0,$$

which is not possible.

Hence we can state the following theorem.

Theorem 3.2. If a Riemannian manifold of dimension $n(n \geq 2)$ admits a semi-symmetric non-metric connection $\bar{\nabla}$ whose Ricci tensor is skew-symmetric and if U is a killing vector field then the manifold can not be Ricci-flat.

Again, for $n = 1$, from (3.7) we see that if $\bar{R}ic(X, Y)$ is skew-symmetric, then Ric(X, Y) = 0 and conversely.

Hence we can state the following theorem.

Theorem 3.3. If a Riemannian manifold of dimension 1 admits a semi-symmetric non-metric connection $\overline{\nabla}$ and if U is a killing vector field, then a necessary and sufficient condition for the Ricci tensor of $\overline{\nabla}$ to be skew-symmetric is that the manifold is Ricci-flat.

4. CONFORMAL CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH RESPECT TO SEMI-SYMMETRIC NON-METRIC CONNECTION

Conformal curvature tensor of a Riemannian manifold with respect to Riemannian connection is given by

$$\begin{aligned}
{}'C(X, Y, Z, W) &= {}'R(X, Y, Z, W) - \frac{1}{n-2}[\mathrm{Ric}(Y, Z)g(X, W) \\
&- \mathrm{Ric}(X, Z)\, g(Y, W) + \mathrm{Ric}(X, W)g(Y, Z) - \mathrm{Ric}(Y, W)g(X, Z)] \\
&+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \qquad (4.1)
\end{aligned}$$

Analogous to this definition, we define conformal curvature tensor of M^n with respect to semi-symmetric non-metric connection given by

$$\begin{aligned}
{}'\overline{C}(X, Y, Z, W) &= {}'\overline{R}(X, Y, Z, W) - \frac{1}{n-2}[\overline{\mathrm{R}}\mathrm{ic}(Y, Z)g(X, W) \\
&- \overline{\mathrm{R}}\mathrm{ic}(X, Z)g(Y, W) + \overline{\mathrm{R}}\mathrm{ic}(X, W)g(Y, Z) - \overline{\mathrm{R}}\mathrm{ic}(Y, W)g(X, Z)] \\
&+ \frac{\overline{r}}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \qquad (4.2)
\end{aligned}$$

where

$${}'C(X, Y, Z, W) = g(C(X, Y)Z, W) \qquad (4.3)$$

and $${}'\overline{C}(X, Y, Z, W) = g(\overline{C}(X, Y)Z, W). \qquad (4.4)$$

Inconsequence of (1.6), (1.9), (1.10), (2.10), (4.1) and (4.2), we find

$$\begin{aligned}
{}'\overline{C}(X, Y, Z, W) &= {}'C(X, Y, Z, W) + da(X, Y)g(Z, W) \\
&+ \frac{1}{n-2}[da(Y, Z)g(X, W) - da(X, Z)g(Y, W) \\
&+ da(X, W)g(Y, Z) - da(Y, W)g(X, Z)] \\
&+ \frac{1}{n-2}[\alpha(Y, Z)g(X, W) - \alpha(X, Z)g(Y, W)]
\end{aligned}$$

$$-\frac{n-1}{n-2}[\alpha(Y, W)g(X, Z) - \alpha(X, W)g(Y, Z)]$$

$$-\frac{p}{n-2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \tag{4.5}$$

If $'\bar{C}(X, Y, Z, W) = {}'C(X, Y, Z, W)$, then from (4.5) we have

$$\begin{aligned}&+ da(X, Y)g(Z, W) + \frac{1}{n-2}[da(Y, Z)g(X, W) - da(X, Z)g(Y, W)\\&+ da(X, W)g(Y, Z) - da(Y, W)g(X, Z)] + \frac{1}{n-2}[\alpha(Y, Z)g(X, W)\\&- \alpha(X, Z)g(Y, W)] - \frac{n-1}{n-2}[\alpha(Y, W)g(X, Z) - \alpha(X, W)g(Y, Z)]\\&- \frac{p}{n-2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] = 0.\end{aligned} \tag{4.6}$$

Contracting for Z and W, we get from (4.6)

$$du(X, Y) + nda(X, Y) = 0. \tag{4.7}$$

Hence we can state the following theorem.

Theorem 4.1. A necessary condition for the conformal curvature tensor C of the manifold M^n with respect to Levi-Civita connection and conformal curvature tensor $\bar{C}$ of the manifold M^n with respect to semi-symmetric non-metric connection to be equal is that

$$du(X, Y) + nda(X, Y) = 0.$$

Let $\bar{R}(X, Y)Z = 0$ (4.8)

which gives

$$\bar{R}ic(Y, Z) = 0 \tag{4.9}$$

and $\bar{r} = 0.$ (4.10)

Inconsequence of (2.15), (4.2), (4.5), (4.8), (4.9) and (4.10), we have

$$\begin{aligned}'C(X, Y, Z, W) = &-da(X, Y)g(Z, W) - \frac{1}{n-2}[da(Y, Z)g(X, W)\\&- da(X, Z)g(Y, W) + da(X, W)g(Y, Z) - da(Y, W)g(X, Z)]\\&- \frac{1}{(n-1)(n-2)}[Ric(Y, Z)\, g(X, W) - Ric(X, Z)g(Y, W)\\&- da(Y, Z)g(X, W) + da(X, Z)g(Y, W)]\end{aligned}$$

$$+\frac{1}{n-2}[\mathrm{Ric}(Y, W)g(X, Z) - \mathrm{Ric}(X, W)g(Y, Z)$$
$$- da(Y, W)g(X, Z) + da(X, W)g(Y, Z)]$$
$$+\frac{p}{n-2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \tag{4.11}$$

Also, if $'C(X, Y, Z, W) = 0$, then from (4.11), we get

$$- da(X, Y)g(Z, W) - \frac{1}{n-2}[da(Y, Z)g(X, W) - da(X, Z)g(Y, W)$$
$$+ da(X, W)g(Y, Z) - da(Y, W)g(X, Z)] - \frac{1}{(n-1)(n-2)}[\mathrm{Ric}(Y, Z)g(X, W)$$
$$- \mathrm{Ric}(X, Z)g(Y, W) - da(Y, Z)g(X, W) + da(X, Z)g(Y, W)]$$
$$+\frac{1}{n-2}[\mathrm{Ric}(Y, W)g(X, Z) - \mathrm{Ric}(X, W)g(Y, Z)$$
$$- da(Y, W)g(X, Z) + da(X, W)g(Y, Z)] + \frac{p}{n-2}[g(Y, Z)g(X, W)$$
$$- g(X, Z)g(Y, W) = 0.$$

Contracting above for Z and W, we get

$$da(X, Y) = 0. \qquad \text{for } n \geq 3$$

Hence we have the following theorem.

Theorem 4.2. If a Riemannian manifold M^n $(n \geq 3)$ with vanishing curvature tensor with respect to semi-symmetric non-metric connection is conformally flat, then 1-form a is closed.

From (4.5), we have

Theorem 4.3. The conformal curvature tensor $\overline{C}$ of the Riemannian manifold with respect to semi-symmetric non-metric connection satisfies following algebraic properties

$$'\overline{C}(X, Y, Z, W) + '\overline{C}(Y, X, Z, W) = 0 \tag{4.12}$$

and

$$'\overline{C}(X, Y, Z, W) + '\overline{C}(Y, Z, X, W) + '\overline{C}(Z, X, Y, W)$$
$$= \frac{n}{n-2}[da(X, Y)g(Z, W) + da(Y, Z)g(X, W) + da(Z, X)g(Y, W)]$$
$$+ \frac{1}{n-2}[du(X, Y)g(Z, W) + du(Y, Z)g(X, W) + du(Z, X)g(Y, W)]. \tag{4.13}$$

5. CONCIRCULAR CURVATURE TENSOR OF A RIEMANNIAN MANIFOLD WITH RESPECT TO SEMI-SYMMETRIC NON-METRIC CONNECTION

Concricular curvature tensor of a Riemannian manifold with respect to Riemannian connection is given by

$$L(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{5.1}$$

Analogous to this definition, we define concircular curvature tensor of a Riemannian manifold M^n with respect to semi-symmetric non-metric connection given by

$$\bar{L}(X, Y)Z = \bar{R}(X, Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{5.2}$$

From (1.6), (1.10), (5.1) and (5.2), we have

$$\bar{L}(X, Y)Z = L(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X + da(X, Y)Z$$

$$+ \frac{p}{n}[g(Y, Z)X - g(X, Z)Y]. \tag{5.3}$$

If $\bar{L}(X, Y)Z = L(X, Y)Z$, then from (5.3), we get

$$\alpha(X, Z)Y - (Y, Z)X + da(X, Y)Z$$

$$+ \frac{p}{n}[g(Y, Z)X - g(X, Z)Y] = 0$$

Contracting above with respect to Z, we get

$$du(X, Y) + nda(X, Y) = 0. \tag{5.4}$$

Hence we can state the following theorem.

Theorem 5.1. A necessary condition for the concircular curvature tensor L of the manifold M^n with respect to the Levi-Civita connection and the concircular curvature tensor $\bar{L}$ of the manifold M^n with respect to semi-symmetric non-metric connection to be equal is that

$$du(X, Y) + nda(X, Y) = 0.$$

In consequences of (2.15), (4.8), (4.9), (4.10), (5.2) and (5.3), we have

$$L(X, Y)Z = \frac{1}{n-1}[Ric(Y, Z)X - Ric(X, Z)Y]$$

$$- \frac{1}{n-1}[da(Y, Z)X - da(X, Z)Y] - da(X, Y)Z$$

$$-\frac{p}{n}[g(Y, Z)X - g(X, Z)Y]. \tag{5.5}$$

Also, if $L(X, Y)Z = 0$, then from (5.5), we get

$$\frac{1}{n-1}[Ric(Y, Z)X - Ric(X,Z)Y] - \frac{1}{n-1}[da(Y, Z)X - da(X, Z)Y]$$

$$- da(X, Y)Z - \frac{p}{n}[g(Y,Z)X - g(X, Z)Y] = 0.$$

Contracting above with respect to Z, we get

$$da(X, Y) = 0 \quad \text{for } n \geq 3$$

Hence we have the following theorem.

Theorem 5.2. If a Riemannian manifold M^n ($n \geq 3$) with vanishing curvature tensor with respect to semi-symmetric non-metric connection is concircularly flat, then 1-form a closed.

From (5.3), we have the following statement.

Theorem 5.3. The concircular curvature tensor of a Riemannian manifold with respect to semi-symmetric non-metric connection satisfies following algebraic properties

$$\bar{L}(X, Y)Z + \bar{L}(Y, X)Z = 0 \tag{5.6}$$

and

$$\begin{aligned} &\bar{L}(X, Y)Z + \bar{L}(Y, Z)X + \bar{L}(Z, X)Y \\ &= du(Z, Y)X + du(X, Z)Y + du(Y, X)Z \\ &+ da(Y, Z)X + da(Z, X)Y + da(X, Y)Z. \end{aligned} \tag{5.7}$$

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Jour. Pure Math., Vol. 23, 2006, pp. 135-150

REPRESENTATION OF P-ADIC INTEGERS AS SEQUENCE SPACE Ω_p AND ITS ALGEBRIC STRUCTURE AS PID

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ABSTRACT : In this paper we have represented the p-adic integers as a sequence space Ω_p and subsequently we have given PID structure to Ω_p. For this purpose, we have introduced the concept of 'Carry-out sequences' and we have developed some interesting properties of them.

1. INTRODUCTION

Kurt Hensel (1861-1941) discovered the p-adic numbers around the end of nineteenth century, where p stands for a fixed prime. In fact, the p-adic numbers penetrated several mathematical fields like number theory, algebraic geometry, algebraic topology, analysis etc. In this paper we have represented the p-adic integers as a sequence space Ω_p and subsequently we have introduced the algebraic structure to Ω_p. Although our work is analogous with Z_p, one will find the beauty of the carry-out sequences and their involvement during the introduction of PID structure to Ω_p. It will be worthwhile to mention here that one can't take care of the carry-out sequences in Z_p whereas the process of carried over is always done during addition, subtraction, multiplication etc. in Z_p.

In section I, we have developed some interesting properties of the carry-out sequences and some of these properties have been utilized to give the algebraic structure to Ω_p. Those results [3.4 – 3.6] of the carry-out sequences developed in section I, which are not utilized in the said purpose, do have their own beauty and we are looking forward to utilize them in near future. In section II, we have given the PID structure to Ω_p. In this section [prop. 4.3] we have shown how one can use the properties of the carry-out sequences in order to determine the existence of the inverse of an p-adic integer in Ω_p.

2. PRELIMINARIES

Definition 2.1. We define a mapping $\phi : Z_p \to \Omega_p$ by.

$$\phi(x) = \{x_i\} \; \forall x = \sum_{i \geq 0} x_i p^i \; Z_p$$

$$\therefore \Omega_p = \left\{x = \{x_i\}_{i=0}^{\infty} : 0 \leq x_i \leq p-1, \; i = 0, 1, 2 \ldots\right\}$$

Definition 2.2. For any $x = \{x_i\} \in \Omega_p$ we denote the weights of x by $\omega(x)$ and we define it by,

$$\omega(x) = m \text{ if and only if } x_i = 0 \ \forall i < m \text{ and } x_m \neq 0$$

Definition 2.3. For any positive integer k, we define a shifting operator $T^k : \Omega_p \to \Omega_p$ by,

$$T^k(x) = \left\{\overbrace{0, 0, \ldots, 0}^{k \text{ times}}, x_0, x_1, x_2, \ldots\right\} \ \forall x = \{x_i\} \in \Omega_p.$$

Definition 2.4. We define an operator $\psi : \Omega_p \to \Omega_p$ such that for any $x \in \Omega_p$ with $\omega(x) = m$

$$\psi(x) = x' \ \{x_i'\}, \text{ where } x_i' = 0 \ \ \forall i < m;\ x_m + x'_m = p;\ x_i + x'_i = p - 1 \ \forall i > m$$

Definition 2.5. For any $x, y \in \Omega_p$ we define their addition by

$$x + y = z, \text{ where } x_{i-1} + y_{i-1} + a_{i-1} = a_i p + z_{i-1} \ \ \forall i = 1, 2.... \text{ with } a_0 = 0.$$

The sequence $a = \{a_i\}$ is said to be the **carry-out sequence** for the addition of $x, y \in \Omega_p$ and we denote it $[x+y]$.

Definition 2.6. For any $x, y \in \Omega_p$ we define their multiplication by,

$$xy = w, \text{ where } \sum_{j+k=i-1} x_j y_k + b_{i-1} = b_i p + w_{i-1} \ \forall i = 1, 2, \ldots; \text{ with } b_0 = 0.$$

The sequence $b = \{b_i\}$ is said to be the **carry-out sequence** for the multiplication of $x, y \in \Omega_p$ and we denote it by $[xy]$

Remark. Following properties are immediate consequence of the definition of carry-out sequences:

(a) $\omega([x + y]) \geq 1$, $\omega([xy]) \geq 1$.

(b) $[x+0] = 0$ and $[x0] = 0$.

(c) For any $x, y, z, w \in \Omega_p$, if $x_i + y_i = z_i + w_i$ $\forall i = 0, 1, 2....$, then $x + y = z + w$ and

$$[x + y] = [z + w]$$

(d) For any $x, y, z, w \in \Omega_p$, if $x_j y_k = z_j w_k$ $\forall j,k = 0, 1, 2, \ldots$, then $xy = zw$ and $[xy] = [zw]$

(e) For any $x, y \in \Omega_p$, $[x+y] = [y+x]$ and $[xy] = [yx]$.

3. SECTION I

In this section we have evaluated some basic properties of the carry-out sequences as defined in 2.5 and 2.6. Subsequently some of these properties will be utilized in order to introduce the algebraic structure to Ω_p as PID.

Proposition 3.1. For any $x. y \in \Omega_p$ the carry-out sequence $[x+y] \in \Omega_2$.

Proof. Let $[x + y] = a = \{a_i\}$. From definition it follows that $a_0 = 0$. If possible let $a_1 \geq 2$. But $0 \leq x_0 + y_0 \leq 2p - 1$ as $x. y \in \Omega_p$. If $x + y = z$, then we have

$$x_0 + y_0 = a_1 p + z_0 \geq 2p + z_0 \geq 2p, \text{ a contradiction as } 0 \leq x_0, y_0 < p$$

$$\therefore\ a_1 = 0 \text{ or } 1.$$

Let $a_i = 0$ or 1 $\forall i < m$.

In case $a_{m-1} = 0$, by the previous argument $a_m = 0$ or 1.

Let $a_{m-1} = 1$. But,

$$x_{m-1} + y_{m-1} + 1 = a_m p + z_{m-1}.$$

If $a_m \geq 2$ then we have,

$$x_{m-1} + y_{m-1} + 1 \geq 2p + z_{m-1} \geq 2p > 2p - 1.$$

This contradicts the fact that, $x_{m-1} + y_{m-1} + 1 \leq 2p - 1$.

$$\therefore\ a_m = 0 \text{ or } 1.$$

By induction on i the result follows,

Proposition 3.2. For any $x, y \in \Omega_p$, $\omega([xy]) > \omega(x) + \omega(y)$.

Proof. Let $\omega(x) = m$ and $\omega(y) = n$.

If $xy = w$ and $[xy] = a$, then from the definition we have,

$$\sum_{j+k=i} x_j y_k + a_i = a_{i+1}p + w_i \quad \forall i = 0, 1, 2,... \tag{1}$$

Clearly $\sum_{j+k=s} x_j y_k = 0$, for any $s < m + n$ and hence

$$a_s = a_{s+1}p + w_s \qquad \forall s < m + n \tag{2}$$

As $a_0 = 0$, from (2) it follows that $a_1 = 0$

Again $a_1 = 0$ gives $a_2 = 0$

Continuing this process we can see that,

$$a_i = 0 \quad \forall i < m + n$$

Taking $s = m + n - 1$ in (2) we get,

$$0 = a_{m+n}P + w_{m+n-1}$$

Hence $a_{m+n} = 0$ and consequently $\omega([xy]) > \omega(x) + \omega(y)$.

Proposition 3.3. For any $x, y \in \Omega_p$ if $\omega(x) = m$, then $\omega([x + \psi(x)] = m + 1$ and

$$[x + \psi(x)] = \left\{\overbrace{0, 0, \ldots, 0}^{m+1 \text{ times}}, 1, 1, 1, \ldots\right\}$$

Proof. If $\psi(x) = x'$, then by the definition **2.4** we have,

$$x_i' = 0 \ \forall i < m; \ x_m + x_m' = p \text{ and } x_i + x'_i = p - 1 \quad \forall i > m.$$

Let $x + x' = z$. Then we have,

$$x_i + x'_i + a_i = a_{i+1}p + z_i \quad \forall i = 0, 1, 2, \ldots \text{ with } a_0 = 0 \tag{1}$$

where $[x + \psi(x)] = a$.

From (1) it follows that, $a_i = 0 \ \ \forall i \leq m$, for $x_i = x'_i = 0 \quad \forall i < m$

Taking $i = m$ in (1) we get,

$$p = a_{m+1}p + z_m \text{ which implies } a_{m+1} = 1.$$

Hence $\omega([x + \psi(x)]) = m + 1$.

Again taking $i = m + 1$ in (1)

$$p - 1 + 1 = a_{m+2}p + z_{m+1} \text{ which implies } a_{m+2} = 1.$$

Continuing this process it can be seen that $a_i = 1 \quad \forall i > m$

This completes the proof.

Proposition 3.4. For any $x, y \in \Omega_p$, if $\omega(x) = \omega(y) = m$, $[x + y] = a$, $[\psi(x) + \psi(y)] = b$ and $x_m + y_m = p$, then

(i) $\omega(a) = \omega(b) = m + 1$ and $\omega([a + b]) = m + 2$

(ii) $a + b = 0$ and $[a + b] = \left\{\overbrace{0, 0, 0, \ldots, 0}^{(m+2) \text{ times}}, 1, 1, 1, \ldots\right\}$

provided $\omega(x + y) = m + 1$.

Proof. If $\psi(x) = x'$ and $\psi(y) = y'$, then

$x + y = z \Rightarrow x_{i-1} + y_{i-1} + a_{i-1} = a_i p + z_{i-1} \quad \forall i = 1, 2, \ldots;$ with $a_0 = 0$ (1)

$x' + y' = w \Rightarrow x'_{i-1} + y'_{i-1} + b_{i-1} = b_i p + w_{i-1} \quad \forall i = 1, 2, \ldots;$ with $b_0 = 0$ (2)

$\therefore (x_{i-1} + x'_{i-1}) + (y_{i-1} + y'_{i-1}) + (a_{i-1} + b_{i-1}) = (a_i + b_i)p + (z_{i-1} + w_{i-1}), \quad \forall i = 1, 2 \ldots$ (3)

Since $x_{i-1} + x'_{i-1} + y_{i-1} + y'_{i-1} = 0 \quad \forall i < m$, we obtain

$a_i = 0 = b_i \quad \forall i \leq m$ and $z_i = 0 = w_i \quad \forall i < m$.

Taking $i = m + 1$ in (1),

$p = a_{m+1}p + z_m \quad [\because x_m + y_m = p]$

which implies $a_{m+1} = 1$ and $z_m = 0$.

This shows that $\omega(a) = m + 1$.

Clearly $x'_m + y'_m = p$ follows from $x_m + x'_m = y_m + y'_m = x_m + y_m = p$.

Hence by the same argument we obtain $\omega(b) = m + 1$ and $b_{m+1} = 1$.

Now taking $i = m + 2$ in (3) we get.

$2p = (a_{m+2} + b_{m+2})p + (z_{m+1} + w_{m+1})$ (4)

$\Rightarrow z_{m+1} + w_{m+1} = (2 - a_{m+2} - b_{m+2})p$

$\Rightarrow z_{m+1} + w_{m+1} = 0$, p or $2p$ $[\because a_i, b_i \ \{0,1\}]$

$\Rightarrow z_{m+1} + w_{m+1} = 0$ or p $[\because z_{m+1}, w_{m+1} < p]$

$\Rightarrow z_{m+1} + w_{m+1} = p \quad [\because \omega(x + y) = m + 1]$.

Thus from (4) we obtain

$a_{m+2} + b_{m+2} = 1$.

Again taking $i = m+3$ in (3) we obtain

$2p - 1 = (a_{m+3} + b_{m+3})p + (z_{m+2} + w_{m+2})$ (5)

$\Rightarrow z_{m+2} + w_{m+2} + 1 = (2 - a_{m+3} + b_{m+3})p$

$\Rightarrow z_{m+2} + w_{m+2} + 1 = 0$, p or $2p$

$\Rightarrow z_{m+2} + w_{m+2} + 1 = p \quad [\because 0 < 1 \leq z_{m+2} + w_{m+2} + 1 \leq 2p - 1 < 2p]$

Hence from (5) we get,

$a_{m+3} + b_{m+3} = 1$.

Continuing in this way we have,

$a_i + b_i = 1 \ \forall i \geq m + 2$.

Thus from the above discussion we can conclude that

$a_i + b_i = 0 \ \forall i \leq m$

$a_{m+1} + b_{m+1} = 2$

and $a_i + b_i = 1 \quad \forall i \geq m + 2$

Let $[a + b] = d$ and $a + b = r$.

$$\therefore \ a_{i-1} + b_{i-1} + d_{i-1} = d_i.2 + r_{i-1} \ \forall i = 1,2, \qquad (6)$$

Since $a_i + b_i = 0 \quad \forall i \leq m$, therefore from (6) we get,

$d_i = 0 \quad \forall i \leq m + 1$ and $r_i = 0 \ \forall i \leq m$

Taking $i = m + 2$ in (6) we get,

$2 = d_{m+2}, 2 + r_{m+1} \ [\because a_{m+1} + b_{m+1} = 2, d_{m+1} = 0]$

$\therefore \ d_{m+2} = 1$ and $r_{m+1} = 0$.

Again taking $i = m + 3$ in (6) we get,

$2 = d_{m+3}.2 + r_{m+2}$

which implies $d_{m+3} = 1$ and $r_{m+2} = 0$

Continuing in this way we get,

$r_i = 0 \quad \forall i = 0, 1, 2, \ldots$

$d_i = 0 \ \forall i \leq m + 1$ and $d_i = 1 \quad \forall i > m + 1$

$\therefore \ \omega([a + b]) = m + 2;\ a + b = 0$ and

$$[a + b] = \left\{ \overbrace{0,0,\ldots\ldots 0}^{(m+2)\text{ times}},1,1,1,\ldots\ldots \right\}$$

Note 1. The condition $\omega(x + y) = m + 1$ in this result can not be weakened. The following example shows that the result fails if $\omega(x + y) \neq m + 1$

Let us take $x, y \in \Omega_5$ given by,

$x = \{2, 1, 3, 4, 3, 2, 0, \ldots\}$

$y = \{3, 3, 1, 1, 0, 3, 2, \ldots\}$

$\therefore \ x + y = \{0, 0, 0, 1, 4, 0, 3, \ldots\}$

$\psi(x) + \psi(y) = \{0, 0, 0, 4, 0, 4, 1, \ldots\}$

$a + b = \{0, 0, 1, 1, 0, 0, 0, 0, \ldots\} \neq 0$

We note that in this case $\omega(x + y) \neq 1$.

Note 2. From the above result we obtain

$a + b = 0$ which implies $\psi(a) = b$ and hence

$\psi([x + y]) = [\psi(x) + \psi(y)]$.

Thus ψ behaves like a group homomorphism under the conditions referred to.

Proposition 3.5. For any $x, y \in \Omega_p$ if $\omega(x) = \omega(y) = m$, $x_m + y_m \neq p$, $[x + y] = a$ and $[\psi(x) + \psi(y)] = b$, then

(i) $[a + b] = 0$

(ii) $\omega(a + b) = m + 1$ and $a + b = \left\{ \overbrace{0,0,\ldots,0}^{(m+1)\text{ times}}, 1,1,1,\ldots \right\}$

Proof. If $\psi(x) = x'$ and $\psi(y) = y'$, then

$$x_{i-1} + x'_{i-1} + y_{i-1} + y'_{i-1} + a_{i-1} + b_{i-1} = (a_i + b_i)p + (z_{i-1} + w_{i-1}) \quad \forall i = 1, 2, \ldots \tag{1}$$

where $x + y = z$ and $x' + y' = w$.

Since $x_i = x'_i = y_i = y'_i = 0 \quad \forall i < m$, from (1) we have,

$a_i = 0 = b_i \ \forall i \leq m$ and $z_i = 0 = w_i \quad \forall i < m$

Taking $i = m + 1$ in (1) we obtain,

$$z_m + w_n = (2 - a_{m+1} - b_{m+1})\, p \tag{2}$$

$\Rightarrow z_m + w_m = 0,\ p$ or $2p$

$\Rightarrow z_m + w_m = 0$ or p $[z_m + w_m < 2p]$.

If $z_m + w_m = 0$, then $a_{m+1} + b_{m+1} = 2$ i.e. $a_{m+1} = 1 = b_{m+1}$.

But then, $x_m + y_m = 1.p + 0$, a contradiction.

$\therefore z_m + w_m = p$

Thus from (2) we have,

$a_{m+1} + b_{m+1} = 1$

Again taking $i = m+2$ in (1) we obtain,

$$2p - 2 + 1 = (a_{m+2} + b_{m+2})p + (z_{m+1} + w_{m+1}) \tag{3}$$

$\Rightarrow z_{m+1} + w_{m+1} + 1 = (2 - a_{m+2} - b_{m+2})p$

$$\Rightarrow z_{m+1} + w_{m+1} + 1 = p \tag{4}$$

Combining (3) and (4) we obtain,

$$a_{m+2} + b_{m+2} = 1.$$

Continuing in this way we obtain,

$$a_i + b_i = 1 \quad \forall i > m.$$

If $[a + b] = d$ and $a + b = r$.

$$a_i + b_i + d_i = d_{i+1}.2 + r_i \quad \forall i = 0, 1, 2, 3, \ldots \tag{5}$$

As $a_i + b_i = 0 \ \forall i \le m$, from (5) we obtain,

$$d_i = 0 \quad \forall i \le m + 1 \text{ and } r_i = 0 \quad \forall i \le m.$$

Taking $i = m + 1$ (5),

$$1 = d_{m+2}.2 + r_{m+1}$$

$$\therefore\ d_{m+2} = 0 \text{ and } r_{m+1} = 1.$$

Continuing in this way we obtain,

$$d_i = 0 \quad \forall i > m + 1 \text{ and } r_i = 1 \quad \forall i > m.$$

Hence, $[a + b] = 0$, $\omega(a + b) = m + 1$ and

$$a + b = \left\{ \overbrace{0,0,\ldots,0}^{(m+1)\text{ times}}, 1,1,\ldots \right\}.$$

Proposition 3.6. For any $x, y \in \Omega_p$ with $\omega(x) \ne \omega(y)$, if $[x + y] = a$, $[\psi(x) + \psi(y)]=b$, then

(i) $[a + b] = 0$

(ii) $\omega(a + b) = \max\{\omega(x), \omega(y)\} + 1 = k + 1$, (say) and

$$a + b = \left\{ \overbrace{0,0,\ldots,0}^{(k+1)\text{ times}}, 1,1,1,\ldots \right\}.$$

Proof. Let $\omega(x) = m$ and $\omega(y) = n$ and without loss of generality we assume that $m > n$ so that,

$$k = \max\{\omega(x), \omega(y)\} = m$$

If $\psi(x) = x'$ and $\psi(y) = y'$, then

$$x_{i-1} + x'_{i-1} + y_{i-1} + y'_{i-1} + a_{i-1} + b_{i-1} = (a_i + b_i)p + (z_{i-1} + w_{i-1}) \quad \forall i = 1, 2, 3, \ldots \text{(1)}$$

where $x + y = z$ and $\psi(x) + \psi(y)] = w$.

Clearly, $a_j + b_j = 0 \quad \forall j \leq n \quad [\because n < m]$.

Taking $i = n + 1$ in (1)

$$p = (a_{n+1} + b_{n+1})p + (z_n + w_n) \tag{2}$$

$$\Rightarrow z_n + w_n = (1 - a_{n+1} - b_{n+1})p$$

$$\Rightarrow z_n + w_n = 0 \text{ or } p \; [\because z_n + w_n < 2p].$$

If $z_n + w_n = 0$, then from $x + y = z$ we obtain, $x_n + y_n = a_{n+1}p$

$$\therefore y_n = 0 \quad [\because x_n = 0].$$

But this is a contradiction, as $y_n \neq 0$.

$$\therefore z_n + w_n = p.$$

Then from (2) we obtain.

$$a_{n+1} + b_{n+1} = 0$$

We now show that,

$$a_i + b_i = 0 \quad \forall i = n + 2, n + 3, ..., m.$$

Taking $i = n + 2$ in (1),

$$p - 1 = (a_{n+2} + b_{n+2})p + (z_{n+1} + w_{n+1})$$

$$\Rightarrow (z_{n+1} + w_{n+1} + 1) = (1 - a_{n+2} - b_{n+2})p$$

$$\Rightarrow z_{n+1} + w_{n+1} + 1 = p \quad [\because 0 < z_{n+1} + w_{n+1} + 1 < 2p]$$

$$\therefore a_{n+2} + b_{n+2} = 0.$$

Continuing in this way upto $i = m - 1$ we obtain,

$$a_i + b_i = 0 \quad \forall i \leq m.$$

Lastly we now show that,

$$a_i + b_i = 1 \quad \forall i \geq m + 1.$$

Taking $i = m + 1$ in (1),

$$p + (p - 1) = (a_{m+1} + b_{m+1})p + (z_m + w_m)$$

$$\Rightarrow z_m + w_m + 1 = p$$

$$\therefore a_{m+1} + b_{m+1} = 1.$$

Continuing in this way we obtain,

$$a_i + b_i = 1 \quad \forall i \geq m + 1.$$

This shows that $\omega(a + b) = m + 1$.

If $a + b = r$ and $[a + b] = d$, then

$$a_i + b_i + d_i = d_{i+1}.2 + r_i \quad \forall i = 0, 1, 2, \ldots \tag{3}$$

$$\therefore\ d_i = 0 \quad \forall i \leq m + 1 \text{ and } r_i = 0 \quad \forall i \leq m \quad [\because\ a_i + b_i = 0 \quad \forall i \leq m]$$

Taking $i = m + 1$ in (3) we get,

$1 = d_{m+2}.2 + r_{m+1}$ which implies

$d_{m+2} = 0$ and $r_{m+1} = 1$.

Continuing in this way we obtain,

$d_i = 0 \quad \forall i > m + 1$ and $r_i = 1 \quad \forall i \geq m + 1$.

This completes the proof.

Proposition. 3.7. For any $x, y, z \in \Omega_p$.

$$[x + y] + [(x + y) + z] = [y + z] + [x + (y + z)]$$

Proof. If $[x + y] = a$, $[(x + y) + z] = b$, $[y + z] = c$ and $[x + (y + z)] = d$, then we have,

$$x + y = w \Rightarrow x_{i-1} + y_{i-1} + a_{i-1} = a_i p + w_{i-1} \quad \forall i = 1, 2, 3, \ldots \text{ with } a_0 = 0 \tag{1}$$

$$(x + y) + z = w' \Rightarrow w_{i-1} + z_{i-1} + b_{i-1} = b_i p + w'_{i-1} \quad \forall i = 1, 2, 3, \ldots \text{ with } b_0 = 0 \tag{2}$$

$$x + z = q \Rightarrow y_{i-1} + z_{i-1} + c_{i-1} = c_i p + q_{i-1} \quad \forall i = 1, 2, 3, \ldots \text{ with } c_0 = 0 \tag{3}$$

$$x + (y + z) = q' \Rightarrow x_{i-1} + q_{i-1} + d_{i-1} = d_i p + q'_{i-1} \quad \forall i = 1, 2, 3, \ldots \text{with } d_0 = 0 \tag{4}$$

Combining (1), (2) and (3), (4) we obtain,

$$x_{i-1} + y_{i-1} + z_{i-1} + (a_{i-1} + b_{i-1}) = (a_i + b_i)p + w'_{i-1} \quad \forall i = 1, 2, 3, \ldots \tag{5}$$

$$x_{i-1} + y_{i-1} + z_{i-1} + (c_{i-1} + d_{i-1}) = (c_i + d_i)p + q'_{i-1} \quad \forall i = 1, 2, 3, \ldots \tag{6}$$

Taking $i = 1$ in (5) and (6),

$w'_0 = \{(c_1 + d_1) - (a_1 + b_1)\}p + q'_0$ which implies $a_1 + b_1 = c_1 + d_1$.

Let us assume that,

$$a_i + b_i = c_i + d_i \quad \forall i < m.$$

Taking $i = m$ in (5) and (6),

$w'_{m-1} = \{(c_m + d_m) - (a_m + b_m)\}p + q'_{m-1}$ which implies $a_m + b_m = c_m + d_m$.

Hence by induction,

$$a_i + b_i = c_i + d_i \qquad \forall i = 1, 2, 3, \ldots$$

$$\therefore\ a + b = c + d.$$

Proposition 3.8. For any x, y, z $\in \Omega_p$ if [xy] = a, [(xy)z] = b, [yz] = c and [x(yz)]=d, then

$$\sum_{r+s=i-1}(x_r c_s - z_r a_s) = b_{i-1} - d_{i-1} \qquad \forall i = 1, 2, 3, \ldots.$$

Proof. If xy = w, (xy)z = w', yz = q and x(yz) = q', then it follows from the definition,

$$\sum_{j+k=i-1} x_j y_k + a_{i-1} = a_i p + w_{i-1} \qquad \forall i = 1, 2, 3, \ldots \text{ with } a_0 = 0 \tag{1}$$

$$\sum_{r+s=i-1} w_r z_s + b_{i-1} = b_i p + w'_{i-1} \qquad \forall i = 1, 2, 3, \ldots \text{ with } b_0 = 0 \tag{2}$$

$$\sum_{j+k=i-1} y_j z_k + c_{i-1} = c_i p + q_{i-1} \qquad \forall i = 1, 2, 3, \ldots. \text{ with } c_0 = 0 \tag{3}$$

$$\sum_{r+s=i-1} x_r q_s + d_{i-1} = d_i p + q'_{i-1} \qquad \forall i = 1, 2, 3, \ldots.. \text{ with } d_0 = 0 \tag{4}$$

Combining (1), (2) and (3), (4) we obtain,

$$w'_{i-1} \equiv \left(\sum_{j+k+s=i-1} x_j y_k z_s + \sum_{r+s=i-1} a_r z_s + b_{i-1} \right) (\text{mod } p) \qquad \forall i = 1, 2, 3, \ldots \tag{5}$$

$$q'_{i-1} \equiv \left(\sum_{j+k+s=i-1} x_j y_k z_s \sum_{r+s=i-1} c_r x_s + d_{i-1} \right) (\text{mod } p) \qquad \forall i = 1, 2, 3, \ldots. \tag{6}$$

Clearly $\sum_{r+s=i-1}(c_r x_s - a_r z_s) = b_{i-1} - d_{i-1}$ when i = 1, for $a_0 = b_0 = c_0 = d_0 = 0$

and hence $w'_0 = q'_0$ (mod p). Thus $w'_0 = q'_0$ $[\because 0 \le w'_0, q'_0 < p]$

For 1 = 2 we have,

$$\sum_{r+s=1}(c_r x_s - a_r z_s) = c_1 x_0 - q_1 z_0$$

$$= \frac{1}{p}[x_0(y_0 z_0 + c_0 - q_0) - z_0(x_0 y_0 + a_0 - w_0)]$$

$$= \frac{1}{p}[w_0 z_0 - q_0 x_0]$$

$$= \frac{1}{p}[(b_1 p + w'_0 - b_0) - (d_1 p + q'_0 - d_0)]$$

$$= b_1 - d_1 \quad [\because w'_0 = q'_0].$$

Assuming the result for i − 1 we have,

$$\sum_{r+s=i}(c_r x_s - a_r z_s) = \frac{1}{p}\left[\sum_{r+s=i-1}(c_r x_s - a_r z_s) + \sum_{r+s=i-1} z_r w_s - \sum_{r+s=i-1} x_r q_s\right]$$

$$= \frac{1}{p}\left[(b_{i-1} - d_{i-1}) + (b_i p + w'_{i-1} - b_{i-1}) - (d_i p + q'_{i-1} - d_{i-1})\right]$$

$$= b_i - d_i.$$

Thus the result follows from induction.

Proposition 3.9. For any x, y, z $\in \Omega_p$ if $[y + z] = a$, $[x(y + z)] = b$, $[xy] = c$, $[xz] = d$ and $[xy + xz] = e$, then

$$\sum_{j+k=i-1} x_j a_k = c_{i-1} + d_{i-1} + e_{i-1} - b_{i-1} \quad \forall i = 1, 2, 3, \ldots.$$

Proof. If we take $x + y = w$, $x(y + z) = w'$, $xy = q$, $xz = q'$ and $xy + zx = r$ then we have,

$$y_{i-1} + z_{i-1} + a_{i-1} = a_i p + w_{i-1} \qquad \forall i = 1, 2, 3, \ldots \text{ with } a_0 = 0 \tag{1}$$

$$\sum_{j+k=i-1} x_j w_k + b_{i-1} = b_i p + w'_{i-1} \qquad \forall i = 1, 2, 3, \ldots \text{ with } b_0 = 0 \tag{2}$$

$$\sum_{j+k=i-1} x_j y_k + c_{i-1} = c_i p + q_{i-1} \qquad \forall i = 1, 2, 3, \ldots \text{with } c_0 = 0 \tag{3}$$

$$\sum_{j+k=i-1} x_j z_k + d_{i-1} = d_i p + q'_{i-1} \qquad \forall i = 1, 2, 3, \ldots \text{ with } d_0 = 0 \tag{4}$$

and $q_{i-1} + q'_{i-1} + e_{i-1} = e_i p + r_{i-1} \qquad \forall i = 1, 2, 3, \ldots$ with $e_0 = 0$ (5)

As $a_0 = b_0 = c_0 = d_0 = e_0 = 0$, the result is obvious for i = 1, and hence $w'_0 = r_0$.

For i = 2 we have,

$$\sum_{j+k=1} x_j a_k = \frac{1}{p} x_0(y_0 + z_0 - w_0) \quad \text{using (1)}$$

$$= \frac{1}{p}\left[(c_1 p + q_0) + (d_1 p + q'_0) - (b_1 p + w'_0)\right] \quad \text{using (2), (3) and (4)}$$

$$= (c_1 + d_1 - b_1) + \frac{1}{p}(q_0 + q'_0 - w'_0)$$

$$= c_1 + d_1 + e_1 - b_1. \qquad \text{using (5)}$$

Assuming the result for (i − 1), we have

$$\sum_{j+k=i} x_j a_k = \frac{1}{p}\left[\sum_{j+k=i} x_j(y_{k-1} + z_{k-1} + a_{k-1} - w_{k-1})\right]$$

$$= \frac{1}{p}\left[\sum_{j+k=i-1} x_j(y_k + z_k + a_k - w_k)\right]$$

$$= \frac{1}{p}[(c_i p + q_{i-1} - c_{i-1}) + (d_i p + q'_{i-1} - d_{i-1}) + (c_{i-1} + d_{i-1} + e_{i-1} - b_{i-1})$$
$$- (b_i p + w'_{i-1} - b_{i-1})]$$

$$= c_i + d_i + e_i - b_i$$

The result follows from induction.

4. SECTION II

In this section we have shown that Ω_p is a PID. Moreover we have used the concept of the carry-out sequences to determine the invertible elements of Ω_p.

Proposition 4.1. Ω_p is an integral domain under the addition and multiplication as defined in 2.5 and 2.6.

Proof. Obviously Ω_p is closed under addition and multiplication as defined in 2.5 and 2.6. Commutativity in Ω_p w.r.t. addition and multiplication follows from (c) and (d) in the Remark earlier. Moreover 0 = {0, 0, 0,...} and e = {1, 0, 0,...} will serve as the zero element and identity element in Ω_p. For any x ($\neq 0$) $\in \Omega_p$ if $\omega(x) = m$ and $[x + \psi(x)] = a$, then by proposition 3.3,

$$\omega(a) = m + 1 \text{ and } a_i = 1 \quad \forall i \geq m + 1$$

If we take $\psi(x) = x'$ and $x + x' = z$, then

$$x_i + x'_i + a_i = a_{i+1}.p + z_i \quad \forall i = 0, 1, 2, \ldots \qquad (1)$$

For $i < m$, $x_i + x'_i = 0 = a_i$, which implies that $z_i = 0$.

For $i = m$, $x_m + x'_m = p$ and $a_m = 0$, which implies that $z_m = 0$.

For $i > m$, $x_i + x'_i = p - 1$ and $a_i = 1$, which implies that $z_i = 0$.

So $x_i + x'_i = 0$, which assure the existence of the negative of any element in Ω_p.

For any $x, y, z \in \Omega_p$ if $[x + y] = a$, $[(x + y) + z] = b$, $[y + z] = c$ and $[x + (y + z)] = d$ then by proposition 3.7 $a + b = c + d$.

If $(x + y) + z = w$ and $x + (y + z) = w'$ then

$$w_i - w'_i = \{(a_i + b_i) - (c_i + d_i)\} - \{(a_{i+1} + b_{i+1}) - (c_{i+1} + d_{i+1})\}p \quad \forall i = 0, 1, 2....$$

$$= 0 \quad \forall i = 0, 1, 2,....$$

This proves associativity in Ω_p under addition.

If $(xy)z = w'$ and $x(yz) = q'$, it follows that

$$\sum_{j+k+s=i-1} (x_j y_k) z_s + \left(\sum_{r+s=i-1} a_r z_s + b_{i-1} \right) = \left(\sum_{r+s=i} a_r z_s + b_i \right) p + w'_{i-1} \quad \forall i = 1, 2, 3,...$$

and $$\sum_{j+k+s=i-1} x_j (y_k z_s) + \left(\sum_{r+s=i-1} c_r x_s + d_{i-1} \right) = \left(\sum_{r+s=i} c_r x_s + d_i \right) p + q'_{i-1} \quad \forall i = 1, 2, 3,....$$

By Proposition 3.8 it follows that $w'_{i-1} = q'_{i-1}$ $\forall i = 1, 2, 3,...$ i.e. $(xy)z = x(yz)$.

Let $x, y, z \in \Omega_p$ such that $xy = 0$. If possible, let neither x nor y is zero and $\omega(x) = m$, $\omega(y) = n$. As $xy = 0$,

$$\sum_{j+k=m+n} x_j y_k + a_{m+n} - a_{m+n+1} \cdot p = 0 \text{ where } [xy] = a$$

$$\Rightarrow x_m y_n = a_{m+n+1} \cdot p \quad [\because \omega([xy]) > m + n]$$

$$\Rightarrow x_m y_n \equiv 0 \pmod p$$

$$\Rightarrow x_m \equiv 0 \pmod p \text{ or } y_n \equiv 0 \pmod p.$$

$$\Rightarrow x_m = 0 \text{ or } y_m = 0, \text{ a contradiction.}$$

So, Ω_p is without zero divisor and hence Ω_p is an integral domain.

Proposition 4.2. Ω_p is PID.

Proof. From the definition 2.3 we have,

$$T^k(\Omega_p) = \{x \in \Omega_p | \omega(x) \geq k\}.$$

As $0 \in T^k(\Omega_p)$, $T^k(\Omega_p) \neq \phi$.

For any $x, y \in T^k(\Omega_p)$ and $r \in \Omega_p$ we have,

$\omega(x - y) = \omega(x + \psi(y)) \geq \omega(x) + \omega(\psi(y)) = \omega(x) + \omega(y) \geq 2k > k.$

Hence $x - y \in \Omega_p$.

In case $x = 0$ or $r = 0$, $rx \in T^k(\Omega_p)$.

Let us assume that neither x nor r is zero and $\omega(x) = m \geq k$, $\omega(r) = n$.

If $rx = z$, then

$$\sum_{j+l=i} x_j r_l + a_i = a_{i+1}p + z_i \qquad \forall i = 0, 1, 2,\ldots \text{ where } [rx] = a$$

$$\therefore\ a_i = a_{i+1}p + z_i \quad \forall i < m + n.$$

As $\omega([xy]) > m + n$, it follows that,

$z_i = 0 \quad \forall i < m + n$.

Hence $\omega(z) \geq m + n > k$ and therefore $rx \in T^k(\Omega_p)$.

Thus $T^k(\Omega_p)$ is a proper ideal of Ω_p, $\forall k = 1, 2, 3,\ldots$

Obviously,

$$\Omega_p \supset T(\Omega_p) \supset T^2(\Omega_p) \supset \ldots\supset \bigcap_{k\geq 0} T^k(\Omega_p) = \{0\}.$$

For any $x \in T^k(\Omega_p)$, we take $y = \{x_k, x_{k+1},\ldots\} \in \Omega_p$ so that

$$[T^k(e)]y = x.$$

So $T^k(\Omega_p)$ is a principal ideal generated by $T^k(e)$.

Let us choose a nonzero proper ideal I of Ω_p and let $x\ (\neq 0) \in I$.

We now consider the set A given by,

$$A = \{\omega(x) \mid x(\neq 0) \in I\} \neq \phi.$$

Let m be the least element of A.

Now, $y \in I$ and $\omega(y) = n \Rightarrow m \leq n$

$$\Rightarrow T^m(\Omega_p) \supset T^n(\Omega_p).$$

$$\Rightarrow y \in T^m(\Omega_p).$$

This proves that $I \subseteq T^m(\Omega_p)$. (1)

For $a(\neq 0) \in I$ with $\omega(a) = m$ we consider $u = \{a_m, a_{m+1},\ldots\}$ so that

$$T^m(e)u = a$$

$$\Rightarrow T^m(e) = au^{-1} \in I.$$

So, $T^m(\Omega_p) \subseteq I$ (2)

From (1) and (2) we obtain,

$$I = T^m(\Omega_p).$$

Hence Ω_p is a PID.

We conclude this paper with the following necessary and sufficient condition for the invertibility of a nonzero element in Ω_p.

Proposition 4.3. An element $x\ (\neq 0) \in \Omega_p$ is invertible if and only if $\omega([x + \psi(x)])=1$

Proof. Let x be invertible. Then $x_0 \neq 0$ and hence $\omega(x) = 0$.

$$\therefore\ \omega([x + \psi(x)]) = 1 \qquad \text{[By Prop. 3.3]}$$

Conversely, let $\omega(x + \psi(x)] = 1$ and $[x + \psi(x)] = a$.

If $\psi(x) = x'$,

$$x_i + x'_i + a_i = a_{i+1}p + r_i \quad \forall i = 0, 1, 2, 3,...$$

$$\therefore\ x_0 + x'_0 = p + r_i \quad [\because a_0 = 0, a_1 = 1]$$

$$\Rightarrow x_0 + x'_0 \geq p$$

$$\Rightarrow x_0 \neq 0$$

Hence x is invertible.

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Jour. Pure Math., Vol. 23, 2006, pp. 151-158

RELATIVE ORDER OF AN ENTIRE FUNCTION

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ABSTRACT : Following Bernal [1] we consider relative order ρ_g (f) of an entire function f relative to the entire functiong g. We have obtained a relation between ρ_g and $n_f(r)$, the number of zeros of f in $\overline{B}(0,r)$. We have also obtained a few relations between ρ_g and $\mu_f(r)$, $\nu_f(r)$ where $\mu_f(r)$ is the maximum term of the power series representing f for $|z| = r$ and $\nu_f(r)$ is the rank of the maximum term $\mu_f(r)$. Finally, we have shown that $\rho_g(f_1 \pm f_2) = \max\{\rho_g(f_1), \rho_g(f_2)\}$ where $\rho_g(f_1) \neq \rho_g(f_2)$.

Key words : Relative order, maximum term, rank of a maximum term.

A.M.S. Classification : 30D.

1. INTRODUCTION

Let f be a transcendental entire function and $M_f(r) = \max_{|z|=r} |f(z)|$. It is well known that $M_f(r)$ increases infinitely faster than $M_p(r)$, where p is any polynomial; that is,

$$\lim_{r\to\infty} \frac{M_f(r)}{M_p(r)} = \infty$$

So, to measure the growth-rate of an entire function, as determined by its order, the conventional scale is the exponential function. That is, the growth-rate of an entire function f is measured relative to the growth-rate of the exponential function : exp (z). The problem is : relative to this scale, every polynomial is of order zero, that is, all the polynomials have the same growth-rate even though a polynomial of higher degree grows faster than a polynomial of lower degree.

The solve this problem, Berna [1] took an arbitrary, but fixed, entire function, say $g(z)$, as the measuring scale. He then defined relative order of an entire function f with respect to g. Now different polynomials may have different growth-rate with respect to g, if we take a fixed polynomial as g. However, if $g(z) = \exp(z)$, all the classical results on the growth of an entire function will follow.

2. DEFINITIONS AND NOTATIONS

Let f, g be non-constant entire functions. Then, $M_f(r)$, $M_g(r)$ are strictly increasing, increase to ∞, and continuous functions of r and the inverse function $M_f^{-1}(r) : (|f(0)|,\infty) \to (0,\infty)$ exists and $\lim\limits_{r\to\infty} M_f^{-1}(r) = \infty$. Bernal [1] defined the relative order of f with respect to g, denoted ρ_g, as follows :

$$\rho_g = \rho_g(f) = \inf\{k>0 : M_f(r) < M_g(r^k) \text{ for all } r > r_0(k)\}.$$

It easily follows that $\rho_g(f) = \limsup\limits_{r\to\infty} \dfrac{\log M_g^{-1}(M_f(r))}{\log r}$.

From the definition it immediately follows that

(i) If f be a transcendental entire function and g be a polynomial, then $\rho_g(f) = \infty$.

(ii) If f be a polynomial and g be a transcendental entire function, then $\rho_g(f) = 0$.

To exclude these two extreme cases, it is desirable to take the measuring scale g as a polynomial when we measure the growth of a polynomial and g as a transcendental entire function when we measure the growth of a transcendental entire function.

§ 3.

Theorem 3.1. Let f be an entire function with finite relative order ρ_g having infinite number of zeros such that $f(0) \neq 0$. If $n_f(r)$ denotes the number of zeros of f in $\overline{B}(0;r)$, then given $\varepsilon > 0$, there exists r_0 such that for $r>r_0$.

$$n_f\left(\frac{r}{k}\right) \le \frac{1}{\log(k-1)}\left[\log\left(M_g\left(r^{\rho_g+\varepsilon}\right)\right) - \log|f(0)|\right], \text{ where } k > 2.$$

Proof. We take an arbitrary positive number r and let $\alpha_1, \ldots\ldots\ldots\ldots, \alpha_m$ be the zeros of f in $\overline{B}\left(0;\frac{r}{k}\right)$. Let $\phi(z) = \dfrac{f(z)}{\left(1-\frac{z}{\alpha_1}\right)\cdots\left(1-\frac{z}{\alpha_m}\right)}$.

Then $\phi(0) = f(0) \neq 0$. For $|z| = r$, we have, $\left|\frac{z}{\alpha_m}\right| \ge k$ and $|\phi(z)| \le \dfrac{M_f(r)}{(k-1)^m}$. By maximum modulus theorem, $|\phi(0)| \le \dfrac{M_f(r)}{(k-1)^m}$ and so $(k-1)^m \le \dfrac{M_f(r)}{|f(0)|}$.

Hence,

$$m = n_f\left(\frac{r}{k}\right) \le \frac{1}{\log(k-1)}\ \{\log M_f\,(r) - \log |f(0)|\}................(1)$$

Since f is of relative order ρ_g, for a given $\varepsilon > 0$ there exists a positive number r_0 such that $M_f\,(r) < M_g\,\left(r^{\rho_g+\varepsilon}\right)$ for all $r > r_0$. From (1) it follows,

$$n_f\left(\frac{r}{k}\right) \le \frac{1}{\log(k-1)}\{\log M_g\,\left(r^{\rho_g+\varepsilon}\right) - |f(0)|\},\ \text{for } r>r_0 \text{ and } k>2.$$

Corollary. If f be an entire function with finite relative order, ρ_g, then for $\varepsilon > 0$,

$$n_f\left(\frac{r}{2}\right) = 0\left(\log M_g\left(r^{\rho_g+\varepsilon}\right)\right)$$

Theorem 3.2. If f be an entire function with finite relative order ρ_g, then

$$\rho_g \le \limsup_{r\to\infty}\frac{\log M_g^{-1}\left(r^{\frac{rM_{f^1}(r)}{M_f(r)}}\right)}{\log r}.$$

Proof. T. Vijayaraghavan [4] prove that $M_{f^1}(r) \ge \dfrac{\log M_f(r)}{\log r}\cdot\dfrac{M_f(r)}{r}$.

That is, $M_f(r) \le r^{\frac{rM_{f^1}(r)}{M_f(r)}}$. Since M_g^{-1} is an increasing function, we have,

$$\limsup_{r\to\infty}\frac{\log M_g^{-1}\left(r^{\frac{rM_{f^1}(r)}{M_f(r)}}\right)}{\log r} \ge \limsup_{r\to\infty}\frac{\log M_g^{-1}(M_f(r))}{\log r} = \rho_g.$$

§ 4.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ .... (1), be an entire function. The mixmum term $\mu_f(r)$ of the series (1) for $|z| = r$ is defined as

$$\mu_f(r) = \max_{n \geq 0} |a_n| r^n \qquad \ldots(2)$$

The rank $\upsilon_f(r)$ of the maximum term $\mu_f(r)$ is defined as :

$\upsilon_f(r) = \max \{n : \mu_f(r) = |a_n| r^n\}$.

Theorem 4.1. If f be an entire function with finite relative order ρ_g, then

$$\rho_g \leq \limsup_{r \to \infty} \frac{\log M_g^{-1}\left(r^{\nu_f(r)}\right)}{\log r} \leq 2\rho_g$$

where $\upsilon_f(r)$ is the rank of the maximum term $\mu_f(r)$.

Proof. From theorem 3.2 we have,

$$\rho_g \leq \limsup_{r \to \infty} \frac{\log M_g^{-1}\left(r^{\frac{rM_f'(r)}{M_f(r)}}\right)}{\log r}$$

Valiron [3] had shown that $rM_f'(r) \sim \upsilon_f(r)\, M_f(r)$. Hence,

$$\rho_g \leq \limsup_{r \to \infty} \frac{\log M_g^{-1}\left(r^{\nu_f(r)}\right)}{\log r} \qquad \ldots\ldots (1)$$

Again, from the definition of ρ_g, $M_f(r) < M_g\left(r^{\rho_g+\varepsilon}\right)$ for $\varepsilon > 0$ and $r \geq r_0$.

Hence $M_g^{-1}(M_f(r)) < r^{\rho_g+\varepsilon}$. But from the equation [3],

$\log \mu_f(r) = \log \mu_f(r_0) + \int_{r_0}^{r} \frac{\nu_f(x)}{x} dx$, we have,

$$\nu_f(r) \log r \leq \int_{r}^{r^2} \frac{\nu_f(x)}{x} dx \leq \int_{0}^{r^2} \frac{\nu_f(x)}{x} dx = \log \mu_f(r^2).$$

Hence, $r^{\nu_f(r)} \leq \mu_f(r^2)$, that is, $\log M_g^{-1}\left(r^{\nu_f(r)}\right) \leq \log M_g^{-1}(\mu_f(r^2))$.

Since $\log \mu_f(r) \sim \log M_f(r)$, $\log M_g^{-1}\left(r^{\nu_f(r)}\right) \leq \log M_g^{-1}(M_f(r^2))$.

So, $\log M_g^{-1}\left(r^{\nu_f(r)}\right) < 2(\rho_g+\varepsilon) \log r$. Hence,

$$\limsup_{r\to\infty} \frac{\log M_g^{-1}\left(r^{\nu_f(r)}\right)}{\log r} \leq 2\rho_g \qquad .....(2)$$

combining (1) and (2) we get the result.

Corollary 1.

$$\rho_g \leq \limsup_{r\to\infty} \frac{\log M\left(r^{\frac{rM_f{}^1(r)}{M_f(r)}}\right)}{\log r} \leq 2\rho_g.$$

Proof. The corollary follows from the relation $rM_f(r) \sim \nu_f(r)\, M_f(r)$.

Corollary 2.

$$\rho_g \leq \limsup_{r\to\infty} \frac{\log M_g^{-1}\left(r^{\frac{r\mu_f{}^1(r)}{\mu_f(r)}}\right)}{\log r} \leq 2\rho_g.$$

Proof. Let $\upsilon_{f1}(r)$ be the rank of $\mu_f(r)$. We have, [2],

$\mu_{f^1}(r) \geq \frac{\nu_f(r)}{r} . \mu_f(r)$. corollary 2 now follows.

Theorem 4.2. If f and g are non-constant polynomials of degree m and n respectively, then the relative order of f with respect to g, $\rho_g(f) = \frac{m}{n}$.

Proof. Let $f(z) = a_0 + a_1 z + \ldots\ldots\ldots\ldots + a_m z^m$, $a_m \neq 0$, $m \geq 1$ and

$g(z) = b_0 + b_1 z + \ldots\ldots\ldots\ldots + b_n z^n$, $b_n \neq 0$, $n \geq 1$.

We have, for small $\varepsilon > 0$ and large r(=|z|).

$$|a_m|r^m (1-\varepsilon) \le M_f(r) \le |a_m|r^m (1+\varepsilon) \quad \ldots\ldots (1)$$

and $|b_n|r^n (1-\varepsilon) \le M_g(r) \le |b_n|r^n (1+\varepsilon)$ .... (2)

From (2), $M_g^{-1}(|b_n|r^n (1-\varepsilon) \le r$. Taking $|b_n|r^n (1-\varepsilon) = t$ we get.

$$r = \left(\frac{t}{|b_n|(1-\varepsilon)}\right)^{1/n} \cdot \text{ Hence, } M_g^{-1}(r) \le \left(\frac{r}{|b_n|(1-\varepsilon)}\right)^{1/n}.$$

$$\text{Now, } \rho_g(f) = \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log r} \le \limsup_{r\to\infty} \frac{\log M_g^{-1}(|a_m|r^m(1+\varepsilon))}{\log r},$$

Since M_g^{-1} is a monotonic increasing function.

$$\le \limsup_{r\to\infty} \frac{\log\left\{\dfrac{|a_m|r^m(1+\varepsilon)}{|b_n|(1-\varepsilon)}\right\}^{1/n}}{\log r} \le \frac{m}{n} + \limsup_{r\to\infty} \frac{\frac{1}{n}\log\left\{\dfrac{|a_m|(1+\varepsilon)}{|b_n|(1-\varepsilon)}\right\}}{\log r} = \frac{m}{n}.$$

Again, from the inequality $r \le M_g^{-1}(|b_n|r^n(1+\varepsilon))$ we can similarly show that $\rho_g(f) \ge \frac{m}{n}$.

Hence, $\rho_g(f) \ge \frac{m}{n}$.

Corollary 1. $\rho_g(f) = 1$ if and only if f and g have the same degree.

Corollary 2. Let f, g, h be three non-constant polynomials. Then, $\rho_g(f) = \rho_g(h)$ if and only if f and h have the same degree.

To prove the next theorem we need the following lemma :

Lemma [1]. Let f be an entire function, $\alpha > 1$, $0 < \beta < \alpha$. Then, $\beta M_f(r) < M_f(\alpha r)$ for all large r.

Theorem 4.3. Let $\rho_g(f_1)$, $\rho_g(f_2)$ be the relative orders of the entire functions f_1 and f_2 respectively. Then, $\rho_g(f_1 \pm f_2) \le \max(\rho_g(f_1), \rho_g(f_2))$. Moreover, if $\rho_g(f_1) \ne \rho_g(f_2)$, then $\rho_g(f_1 \pm f_2) = \max(\rho_g(f_1), \rho_g(f_2))$

Proof. If at least one of $\rho_g(f_1)$, $\rho_g(f_2)$ is infinite, then the result is obvious. So we assume both of them to be finite and $\rho_g(f_1) \ge \rho_g(f_2)$. Let $\phi = f_1 \pm f_2$.

Now, $M_{f_1}(r) < M_g\left(r^{\rho_g(f_1)+\varepsilon}\right)$ and $M_{f_2}(r) < M_g\left(r^{\rho_g(f_2)+\varepsilon}\right)$ for all $r > r_0$, $\varepsilon > 0$.

So. $M_{\phi}(r) \le M_{f_1}(r) + M_{f_2}(r) < 2M_g(r^{\rho_g(f_1)+\varepsilon}) < M_g(3r^{\rho_g(f_1)+\varepsilon})$ for all large r (by the lemma). Hence, $M_g^{-1}(M_{\phi}(r)) < 3r^{\rho_g(f_1)+\varepsilon}$ and so

$$\limsup_{r\to\infty} \frac{\log M_g^{-1}(M_{\phi}(r))}{\log r} \le \rho_g(f_1) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $\rho_g\,(\phi) \le \rho_g\,(f_1)$ .... (1)

Further, we assume that $\rho_g\,(f_1) > \rho_g\,(f_2)$. There exists a sequence of numbers r_n, tending to infinity such that $M_{f_1}(r_n) > M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right)$ for any $\varepsilon > 0$. Hence

$$M_{\phi}(r_n) \ge M_{f_1}(r_n) - M_{f_2}(r_n) > M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right) - M_g\left(r_n^{\rho_g(f_2)+\varepsilon}\right)$$

$$= M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right)\left\{1 - \frac{M_g\left(r_n^{\rho_g(f_2)+\varepsilon}\right)}{M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right)}\right\}.$$

We can make $\dfrac{M_g\left(r_n^{\rho_g(f_2)+\varepsilon}\right)}{M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right)}$ sufficiently small by choosing ε so small that $\rho_g\,(f_2) + \varepsilon < \rho_g\,(f_1) - \varepsilon$ and n sufficiently large. Then, $M_{\phi}(r_n) > \frac{1}{2} M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right)$, that is $2M_{\phi}(r_n) > M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right)$. By the lemma, $M_{\phi}(3r_n) > M_g\left(r_n^{\rho_g(f_1)-\varepsilon}\right)$ for suffciently large n.

Hence,

$$\limsup_{r\to\infty} \frac{\log M_g^{-1}(M_{\phi}(3r_n))}{\log 3r_n} \rho_g(f_1) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\rho_g\,(\phi) \ge \rho_g\,(f_1)$ .....(2)

From (1) and (2) we have,

$\rho_g\,(f_1) \pm (f_2) = \max(\rho_g\,(f_1), \rho_g\,(f_2))$, when $\rho_g\,(f_1) \ne \rho_g\,(f_2)$.

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JOURNAL OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA

VOLUME 23, 2006

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